What’s Decidable About Program Verification Modulo Axioms?

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We consider the decidability of the verification problem of programs modulo axioms — that is, verifying whether programs satisfy their assertions, when the functions and relations it uses are assumed to interpreted by arbitrary functions and relations that satisfy a set of first-order axioms. Unfortunately, verification of entirely uninterpreted programs (with the empty set of axioms) is already undecidable. A recent work introduced a subclass of coherent uninterpreted programs, and showed that they admit decidable verification [Mathur et al. 2019a]. We undertake a systematic study of various natural axioms for relations and functions, and study the decidability of the coherent verification problem. Axioms include relations being reflexive, symmetric, transitive, or total order relations, functions restricted to being associative, idempotent or commutative, and combinations of such axioms as well. Our comprehensive results unearth a rich landscape that shows that though several axiom classes admit decidability for coherent programs, coherence is not a panacea as several others continue to be undecidable.

CCS Concepts: • Software and its engineering → General programming languages; • Social and professional topics → History of programming languages;

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1 INTRODUCTION

Programs are proved correct against safety specifications typically by induction — the induction hypothesis is specified using inductive invariants of the program, and one proves that the reachable states of the program stays within the region defined by the invariants, inductively. Though there has been tremendous progress in the field of decidable logics for proving invariants are inductive, finding inductive invariants is almost never fully automatic. In essence, completely automated verification of programs is almost always undecidable.

Programs can be viewed as working over a data-domain, with variables storing values over this domain and being updated using constants, functions and relations defined over that domain. Apart from the notable exception of finite data domains, program verification is typically undecidable when the data domain is infinite. In a recent paper, Mathur et. al. [Mathur et al. 2019a] establish new decidability results when the data domain is infinite. Two crucial restrictions are imposed — data domain functions and relations are assumed to be uninterpreted and programs are assumed to be coherent (the meaning of coherence is discussed later in this introduction). The theory of uninterpreted functions is an important theory in SMT solvers that is often used (in conjunction with other theories) to solve feasibility of loop-free program snippets, in bounded model-checking, and to validate verification conditions. The salient aspect of [Mathur et al. 2019a] is to show entire program verification is decidable for the class of coherent programs, without the use of any
inductive invariants (like loop invariants). While the results of [Mathur et al. 2019a] were mainly theoretical, there has been some recent work on applying this theory to verifying memory-safety of heap-manipulating programs [Mathur et al. 2019b].

Data domain functions and relations used in a program, usually have special properties and are not, of course, entirely uninterpreted. The results of [Mathur et al. 2019a] can be seen as an approximate/abstraction-based verification method in practice — if the program verifies assuming functions and relations to be uninterpreted, then the program is correct for any data domain. However, properties of the data domain are often critical in establishing correctness. For example, in order to prove that a sorting program, results in sorted arrays, it is important that the binary relation \(<\) used to compare elements of the array is a total ordering on the underlying data sort. Consequently, constraining the data domain to satisfy certain axioms results in more accurate modeling for verification.

The first question to ask is, perhaps, whether program verification is decidable when the underlying data domain is a particular model (or a complete theory) like arithmetic. However, notice that even when the data domain has only the successor function on natural numbers, programs can compute addition and multiplication (using loops) and hence we can reduce the problem of checking whether Diophantine equations are unsolvable to program verification. Moreover, a reasonable restriction of programs (such as an adaptation of the notion of coherence [Mathur et al. 2019a]) that avoids such a reduction seems hard.

In this paper, we undertake a systematic study of the verification of uninterpreted programs when the data-domains are constrained using theories specified by (universally quantified) axioms. The choice of the axioms we study are guided by two principles. First, we study natural properties of functions and relations. Second, we choose to study axioms that have a decidable quantifier-free fragment of first order logic. The reason is that program executions can easily encode quantifier-free formulae (by computing the terms in variables, and assert Boolean combinations of atomic relations and equality on them). Since we are looking for decidable verification for programs with loops/iteration, it makes little sense to examine axioms where even the quantifier-free fragment is undecidable.

The first axioms on relations we study include axioms of reflexivity, irreflexivity, symmetry, and transitivity, and axioms that define partial and total orders, and combinations of these for different sets of relational symbols. Turning to functions, we study commutativity, associativity, and idempotence, all of which are axioms of the kind \( \forall \overline{x}. t_1 = t_2 \), where \( t_1, t_2 \) are terms built using symbols from the vocabulary and the variables \( \overline{x} \). In general, we study only universally quantified axioms — existential quantification can be handled by Skolemizing and treating the Skolem functions as uninterpreted. Furthermore, apart from the relations and functions constrained using axioms, we also allow other completely uninterpreted functions and relations that the program can manipulate.

**Coherence modulo theories**

Mathur et al [Mathur et al. 2019a] define a subclass of programs, called coherent programs, for which program verification on uninterpreted domains is decidable; without the restriction of coherence, program verification on uninterpreted domains is undecidable. Since our framework is strictly more powerful, we adapt the notion of coherence to incorporate theories. A coherent program [Mathur et al. 2019a] is one where all executions satisfy two properties — memoizing and early-assumes. The memoizing property demands that the program computes any term, modulo congruence induced by the equality assumes in the execution, only once. More precisely, if an execution recomputes a term, the term should be stored in a current variable. The early-assumes restriction demands, intuitively, that whenever the program assumes two terms to be equal, it should do so early, before computing superterms of them.
Table 1. Summary of results of this work on verifying coherent programs modulo several axiom classes, combinations of them, checking coherence modulo axioms, and extensions to $k$-coherence

<table>
<thead>
<tr>
<th>Relational Axioms</th>
<th>Functional Axioms</th>
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<tbody>
<tr>
<td>EPR (Bernays-Scönfinkel)</td>
<td>Undecidable</td>
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<tr>
<td>Reflexivity</td>
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<tr>
<td>Irreflexivity</td>
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<tr>
<td>Symmetry</td>
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<tr>
<td>Transitivity</td>
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<tr>
<td>Strict Partial Order</td>
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<tr>
<td>Strict Total Order</td>
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<tr>
<td>All (non-contradicting) combinations of decidable axioms above</td>
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<td>Checking coherence modulo axioms for all decidable axiom classes above</td>
<td>Decidable</td>
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<tr>
<td>Extension of coherence checking and verification for all decidable axiom classes above to $k$-coherence</td>
<td>Decidable</td>
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</table>

![Table 1](image)

We adapt the above notion to coherence modulo theories\(^1\). The memoizing and early-assumes property are now required modulo the equalities that are entailed by the axioms. More precisely, if the theory is characterized by a set of axioms $\mathcal{A}$, the memoizing property demands that if a program computes a term $t$ and there was another term $t'$ that it had computed earlier which is equivalent to $t$ modulo the assumptions made thus far and the axioms $\mathcal{A}$, then $t'$ must be currently stored in a variable. Similarly, the early-assumes condition is also with respect to the axioms — if the program execution observes a new assumption of equality or a relation holding between terms, then we require that any equality entailed newly by it, the previous assumptions and the axioms $\mathcal{A}$ do not involve a dropped term.

The above is a smooth extension of the notion of coherence defined in [Mathur et al. 2019a]; when $\mathcal{A} = \emptyset$, we retrieve the notion of coherence. Moreover, such an extension is necessary — for coherent programs as defined in [Mathur et al. 2019a] and under most axioms we consider, the program verification problem will be undecidable. From hence on, when we say coherent programs in the presence of axioms, we mean coherence modulo those axioms.

Our aims in this paper are purely theoretical — to establish fundamental results regarding the decidability of coherent program verification under axioms that express a wide variety of natural properties of functions and relations. Exploiting our results by implementing verification engines for appropriate domains is beyond the scope of this paper, and is a task left for future work.

**Main Contributions**

The main contributions of this paper are summarized in Table 1.

First, we study axioms on relations. The EPR (effectively propositional reasoning) [Ramsey 1987] fragment of first order logic is one of the few fragments of first-order logic that is decidable, and has been exploited for bounded model-checking and verification condition validation in the literature [Padon et al. 2016a, 2017, 2016b]. We study axioms written in EPR (i.e., universally

\(^1\)We adapt the definition in a way that preserves the sprit of the definition of coherence. Moreover, if we do not adapt the definition, essentially all axioms classes we study in this paper would be undecidable
quantified formulas involving only relations) and show that (a) verification for even coherent programs, modulo EPR axioms, is undecidable.

We then turn to other particular axioms on relations, and study (b) reflexivity, (c) irreflexivity, and (d) symmetry axioms. We show that verification of coherent programs is decidable even with these axioms. (Note that these axioms can be expressed in the EPR fragment.) These results are proved by augmenting coherent program executions with auxiliary statements that ensure reflexivity/irreflexivity/symmetry while preserving coherence, and by showing that this gives a valid reduction to coherent program verification without axioms, which can then be solved using the streaming algorithm developed in [Mathur et al. 2019a].

We next consider (e) the transitivity axiom, and show that even if some relations are required to be transitive, verification is decidable for coherent programs. Furthermore, we show that (f) combinations of reflexivity, irreflexivity, symmetry, and transitivity, admit a decidable verification problem for coherent program. Using this observation, we conclude decidability when a relation is required to be a strict partial order (irreflexive and transitive) or an equivalence relation.

We then consider (g) axioms that capture total orders and show that they too admit a decidable coherent verification problem. Total orders are also expressible in EPR and in fact their formulation in EPR has been used in program verification, as they can be used in lieu of integers when only ordering is important. For example, they can be used to model data in sorting algorithms, array indices in modeling distributed systems to model process ids and the states of processes, etc. [Padon et al. 2017, 2016b].

Our next set of results consider axioms on functions. Associativity and commutativity are natural and fundamental properties of certain functions (like + and *) and are hence natural to capture/abstract using these axioms. (see [Gulwani and Tiwari 2007] where such abstractions are used in program analysis).

Our first result is that (h) verification of coherent programs is decidable when some functions are assumed to be commutative. This proof works, in a sense, similar to the proofs of reflexivity and symmetric relations in that it augments executions to impose commutativity constraints and we show a reduction to coherent program verification without axioms. We can also show that (i) coherent verification problem is decidable for idempotent functions.

We consider (j) associative functions next. In a result that was surprising to us, we show that, unfortunately, in this case the coherent verification problem is undecidable.

Decidability results outlined above, apply to programs that are coherent modulo the axioms/theories. However, given a program, in order to verify it using our techniques, we also would like to decide whether the program is coherent modulo axioms. We prove (k) that for all the decidable axioms above, checking whether programs are coherent modulo the axioms is a decidable problem. Consequently, under these axioms, we can both check whether programs are coherent modulo the axioms and if they are, verify them.

Mathur et. al. [Mathur et al. 2019a] also introduced the notion of k-coherence, where k ∈ ℕ. An execution is k-coherent if one can use k extra ghost variables and assign values to them to help store terms to satisfy the memoizing property of coherent executions; a program is k-coherent if all its executions are. The authors in [Mathur et al. 2019a] prove that verification of k-coherent programs for (purely) uninterpreted functions/relations is decidable and checking if program is k-coherent is decidable. We can extend both these results for the axiom classes we study — (l) checking k-coherence modulo these theories is decidable and verification of programs that are k-coherent programs modulo these theories is decidable as well.

There are several other results that we mention only in passing. For instance, we show that even for single executions, verifying them modulo equational axioms is undecidable as it is closely related to the word problem for groups. And our positive results for program verification under
axioms for functions (commutativity, idempotence), also shows that bounded model-checking under such axioms is decidable, which can have its own applications.

In summary, we systematically explore natural classes of axioms involving functions and relations, and under a natural extension of coherence that incorporates theories, study the verification problem, uncovering a rich landscape of decidable and undecidable problems, as summarized in Table 1. Technically, we overcome several challenges to prove these results. First, the notion of coherence in [Mathur et al. 2019a] needed to be extended to handle the presence of axioms; and this had to be done carefully to also make sure checking coherence (modulo axioms, in the decidable cases) continues to be decidable. Second, we develop a general program instrumentation technique that encodes certain axioms within executions themselves, and helps in proving decidability results for them. However, handling transitivity axioms and the total order axioms are significantly complex, and the proofs need new automata constructions. The techniques we develop for the various axiom classes however do combine well, and we find, pleasantly, that all combinations of them continue to be decidable.

On the negative side, we find that the restriction to coherent programs, though usually effective in making verification decidable, is not a panacea — there are several simple settings (EPR and associativity axioms), where verification of coherent programs is undecidable.

Due to the large number of results and technically involved proofs, we give only the main theorems and proof gists in the paper; more details can be found in the Appendix.

2 ILLUSTRATIVE EXAMPLE

Consider the problem of searching for an element \( k \) in a sorted list. There are two simple algorithms for this problem. Algorithm 1 (Fig. 1, left) walks through the list from beginning to end, and if it finds \( k \), it sets a Boolean variable \( \text{exists} \) to \( T \) if it does. Notice this algorithm does not exploit the sortedness property of the list. Algorithm 2 (Fig. 1, right) also walks through the list, but it stops as soon as it either finds \( k \) or reaches an element that is larger than \( k \). If it finds the element it sets a Boolean variable \( \text{found} \) to \( T \). If both algorithms are run on the same sorted list, then their answers (namely, \( \text{exists} \) and \( \text{found} \)) must be the same.
assume (T ≠ F);
found := F;
stop := F;
exists := F;
sorted := T;
while (x ≠ NIL) {
    if (stop = F) then {
        if (k = key(x)) then found := T;
        if (k ≤ key(x)) then stop := T;
    }
    if (k = key(x)) then exists := T;
    y := next(x);
    if (y ≠ NIL) then {
        if (k(x) ≠ k(y)) then sorted := F;
    }
    x := y;
}
@post: sorted = T ⇒ found = exists

Fig. 2. Left: Uninterpreted program for finding a key k in a list starting at x with < interpreted as a strict total order. The condition a ≤ b is shorthand for a < b ∨ a = b. Right: A model in which < is not interpreted as a strict total order. The elements in the universe of the model are denoted using circles. Some elements are labeled with variables denoting the initial values of these variables. The edges represent subterm relation. Not all functions are shown in the figure. The model does not satisfy the post-condition on the program on left.

Fig. 2 (on the left) shows a program that weaves the above two algorithms together. The variable x walks down the list using the next pointer. The variable stop is set to T when Algorithm 2 stops searching in the list. The pre condition, namely that the input list is sorted, is captured by tracking another variable sorted whose value is T if consecutive elements are ordered as the list is traversed. The post condition demands that whenever the list is sorted, found and exists be equal when the list has been fully traversed.

The program works on a data domain that provides interpretations for the functions key, next, the initial values of the variables, and the relation <. When < is interpreted to be a strict total order, the program is correct. However, if < is not interpreted as a total order, then the program may be incorrectly deemed as buggy. To see this, consider the data model shown on the right in Fig. 2. The data domain has 9 elements in its universe, with the functions next and key interpreted as shown. Initially, x, y have value e1, NIL is e4, k is e7, T and sorted are e8, and F, found, exists, and stop are e9. The interpretation of < is as follows — e5 < e6, e6 < e7, and e7 < e5. Clearly < is not an order, but the program’s sortedness check “sorted = T” will pass. After the entire list is processed, exists will be set to T when x = e3. On the other hand, stop will be set to T when x = e1 because k = e7 < key(x). Therefore, at the end found = F ≠ exists. The work presented in [Mathur et al. 2019a], where all functions and relations are uninterpreted, would therefore declare this program to be incorrect.

The goal of this paper is to explore several natural restrictions on data models and study the problem of verifying coherent programs for them. When < is constrained to be a total order, the program in Fig. 2 is correct and coherent. Our results (see Section 5.5) show that verification of
such programs when relations are constrained to be strict total orders is decidable, and hence we can build automatic decision procedures that will correctly verify such programs.

3 PRELIMINARIES

We briefly recall the syntax and semantics of uninterpreted programs and the verification problem modulo axioms. Our presentation closely follows the one in [Mathur et al. 2019a].

3.1 Program Syntax

We consider imperative programs with loops over a fixed finite set of variables \( V \) and use constant \((C)\), function \((\mathcal{F})\), and predicate \((\mathcal{R})\) symbols belonging to some first order signature \( \Sigma = (C, \mathcal{F}, \mathcal{R}) \). The grammar for the syntax of programs is given below:

\[
\begin{align*}
\langle \text{stmt} \rangle &::= x := c \mid x := y \mid x := f(z) \mid \text{assume} (\langle \text{cond} \rangle) \mid \text{skip} \mid \langle \text{stmt} \rangle ; \langle \text{stmt} \rangle \\
&\mid \text{while} (\langle \text{cond} \rangle) \langle \text{stmt} \rangle \mid \text{if} (\langle \text{cond} \rangle) \text{then} \langle \text{stmt} \rangle \text{else} \langle \text{stmt} \rangle \\
\langle \text{cond} \rangle &::= x = y \mid x = c \mid c = d \mid R(z) \mid \langle \text{cond} \rangle \lor \langle \text{cond} \rangle \mid \neg \langle \text{cond} \rangle
\end{align*}
\]

Here, \( f \in \mathcal{F}, R \in \mathcal{R}, c, d \in C, x, y \in V, \) and \( z \) is a tuple of variables in \( V \) and constants in \( C \). The syntax allows programs to have assignment statements, conditionals (\text{if-then-else}), looping constructs (\text{while}) and sequencing. Since constants can be modeled using variables that are never re-assigned, we will assume, without loss of generality, that the programs do not use constants. Further, arbitrary Boolean combinations of atomic predicates can be expressed using the \text{if-then-else} construct, and henceforth, we will also assume that all conditionals are atomic (i.e., of the form \( x = y, x \neq y, R(z) \) or \( \neg R(z) \)).

3.2 Executions and Semantics of Uninterpreted Programs

Executions of programs over \( \langle \text{stmt} \rangle \) are words over the following alphabet

\[
\Pi = \{ x := y, x := f(z), \text{assume}(x = y), \text{assume}(x \neq y), \\
\text{assume}(R(z)), \text{assume}(\neg R(z)) \mid x, y, z \text{ are in } V \}
\]

For a program \( s \in \langle \text{stmt} \rangle \), the set of executions of of \( s \), denoted \( \text{Exec}(s) \) is a regular language over the alphabet \( \Pi \) and is given as follows (similar to [Mathur et al. 2019a]).

\[
\begin{align*}
\text{Exec}(\text{skip}) &= \varepsilon \\
\text{Exec}(x := y) &= x := y \\
\text{Exec}(x := f(z)) &= x := f(z) \\
\text{Exec(assume}(c)) &= \text{assume}(c) \\
\text{Exec(if } c \text{ then } s_1 \text{ else } s_2 \text{ )} &= \text{assume}(c) \cdot \text{Exec}(s_1) + \text{assume}(\neg c) \cdot \text{Exec}(s_2) \\
\text{Exec}(s_1 ; s_2) &= \text{Exec}(s_1) \cdot \text{Exec}(s_2) \\
\text{Exec(while } c \text{ } s \text{ )} &= [\text{assume}(c) \cdot \text{Exec}(s_1)] \cdot \text{assume}(\neg c)
\end{align*}
\]

The set of partial executions of \( s \) is the set of prefixes of words in \( \text{Exec}(s) \) and is also regular.

A data model \( M = (U_M, [\cdot]_M) \) for signature \( \Sigma \) is a first order structure with a universe \( U_M \) of elements and interpretations for the constants \( ([c]_M | c \in C) \), functions \( ([f]_M | f \in \mathcal{F}) \) and relations \( ([R]_M | R \in \mathcal{R}) \). Given a first order structure \( M \) over \( \Sigma \) (also referred to as a data model in the rest of the presentation), and an execution \( \rho \in \Pi^* \), the semantics of \( \rho \) on \( M \) is given by \( \text{eval}_M : \Pi^* \times V \rightarrow U_M \) that gives the the valuation of variables in \( V \) at the end of an execution, and is defined as follows. Below, we assume that every variable \( x \in V \) is associated with a designated
constant $\bar{x} \in C$ which denotes its initial value.

\[
\text{eval}_M(e, x) = [\bar{x}]_M \quad \text{for every } x \in V
\]
\[
\text{eval}_M(\rho \cdot "x := y", z) = \text{eval}_M(\rho, y) \quad \text{if } z \text{ is } x
\]
\[
\text{eval}_M(\rho \cdot "x := f(z_1, \ldots, z_r)", y) = [f]_M(\text{eval}_M(\rho, z_1), \ldots, \text{eval}_M(\rho, z_r)) \quad \text{if } y \text{ is } x
\]
\[
\text{eval}_M(\rho \cdot a, x) = \text{eval}_M(\rho, x) \quad \text{otherwise}
\]

**Example 1.** Let us consider the program in Fig. 2. While the program does not strictly obey the syntax of (stmt), it can be easily transformed into one — all statements of the form if (c) then s can be transformed to if (c) then s else skip. Further, complex assume statements like ‘assume($k = \text{key}(x)$)’ can be transformed using additional variables — in this case to ‘$kx := \text{key}(x)$; assume($k = kx$)’, where $kx$ is a new variable.

Now, let us consider the following execution of this program.

\[
\pi = \pi_0 \cdot \text{assume}(x \neq \text{NIL}) \cdot \pi_1 \cdot \text{assume}(x \neq \text{NIL}) \cdot \pi_2 \cdot \text{assume}(x \neq \text{NIL}) \cdot \pi_3 \cdot \text{assume}(x = \text{NIL})
\]

This execution corresponds to entering the loop body exactly three times. $\pi_0$ corresponds to the statements executed prior to entering the loop for the first time, and $\pi_1$, $\pi_2$ and $\pi_3$ correspond to the body of the loop in the first, second and third iteration:

\[
\begin{align*}
\pi_0 &= \text{assume}(T \neq F) \cdot \text{found} := F \cdot \text{stop} := F \cdot \text{exists} := F \cdot \text{sorted} := T \\
\pi_1 &= \text{assume}(\text{stop} = F) \cdot \text{assume}(k \neq \text{key}(x)) \cdot \text{assume}(k < \text{key}(x)) \cdot \text{stop} := T \\
&\quad \cdot \text{assume}(k \neq \text{key}(x)) \cdot y := \text{next}(x) \cdot \text{assume}(y \neq \text{NIL}) \cdot \text{assume}(\text{key}(x) < \text{key}(y)) \cdot x := y \\
\pi_2 &= \text{assume}(\text{stop} \neq F) \cdot \text{assume}(k \neq \text{key}(x)) \cdot y := \text{next}(x) \cdot \text{assume}(y \neq \text{NIL}) \\
\pi_3 &= \text{assume}(\text{stop} \neq F) \cdot \text{assume}(k = \text{key}(x)) \cdot \text{exists} := T \cdot y := \text{next}(x) \cdot \text{assume}(y = \text{NIL}) \cdot x := y
\end{align*}
\]

Now consider the model $M$ shown in Fig. 2 on the right. For this model we have eval$_M(\pi, \text{sorted}) = \text{eval}_M(\pi, \text{exists}) = e_8$, and eval$_M(\pi, \text{found}) = \text{eval}_M(\pi, F) = e_9$.

### 3.3 Feasibility of Executions Modulo Axioms

An execution is said to be feasible in a data model, if every assumption made in the execution, holds on the model. More precisely, an execution $\rho$ is feasible in $M$ if for every prefix $\sigma' = \sigma \cdot \text{assume } c$ of $\rho$, we have (a) eval$_M(\sigma, x) = \text{eval}_M(\sigma, y)$ if $c = (x = y)$, (b) eval$_M(\sigma, x) \neq \text{eval}_M(\sigma, y)$ if $c = (x \neq y)$, (c) eval$_M(\sigma, z_1), \ldots, \text{eval}_M(\sigma, z_r) \in [R]_M$ if $c = R(z_1, \ldots, z_r)$, and (d) eval$_M(\sigma, z_1), \ldots, \text{eval}_M(\sigma, z_r) \notin [R]_M$ if $c = \neg R(z_1, \ldots, z_r)$.

Let $\mathcal{A}$ be a set of first order sentences, including possible ground atomic predicates. We say that a data model $M$ is an $\mathcal{A}$-model, denoted $M \models \mathcal{A}$, if for every $\varphi \in \mathcal{A}$, we have $M \models \varphi$. A formula $\varphi$ is $\mathcal{A}$-valid, denoted $\mathcal{A} \models \varphi$, if $\varphi$ holds in every model $M$ that satisfies $\mathcal{A}$.

An execution $\rho$ is said to be feasible modulo $\mathcal{A}$ if there is an $\mathcal{A}$-model $M$ such that $\rho$ is feasible in $M$.

**Example 2.** Let us again consider the execution $\pi$ from Example 1. We first observe that $\pi$ is feasible on the model $M$ from Fig. 2 (right).

Now let us consider the set of axioms $\mathcal{A}_{\text{STO}}$ that states that the relation symbol $<$ used in the program in Fig. 2 (left) is interpreted to be a strict total order. That is

\[
\mathcal{A}_{\text{STO}} = \{ \forall x. \lnot(x < x), \forall x, y, z. x < y \land y < z \rightarrow x < z, \forall x, y. x = y \lor x < y \lor y < x \}
\]

\[\text{irreflexivity} \quad \text{transitivity} \quad \text{totality}\]

\[\text{A ground atomic predicate is of the form } t_1 \sim t_2, \text{ or } R(t_1, \ldots, t_k) \text{ or } \lnot R(t_1, \ldots, t_k), \text{ where } \sim \in \{=, \neq\}, R \text{ is a relation symbol, and } t_i \text{s are ground terms.}\]
Observe that the model $M$ is not a $\mathcal{A}_{\text{STO}}$-model because there is a cyclic dependency — $e_5 < e_6$, $e_6 < e_7$ and $e_7 < e_5$. Now consider the model $M'$ which differs from $M$ only in the interpretation of $<$ as: $[<]_{M'} = \{(e_5, e_6), (e_6, e_7), (e_5, e_7)\}$. It is easy to see that $M'$ is an $\mathcal{A}_{\text{STO}}$ model and the execution $\pi$ is not feasible on $M'$. In fact, there is no $\mathcal{A}_{\text{STO}}$-model on which $\pi$ is feasible, or, as we say, $\pi$ is infeasible modulo $\mathcal{A}_{\text{STO}}$.

3.4 Program Verification Modulo Axioms

We consider programs annotated with post-conditions that are over the following syntax below. Here, $x$, $y$ and $z$ belong to the set of program variables $V$ and $R \in \mathcal{R}$ is a relation symbol in $\Sigma$.

$$\mathcal{L} : \varphi ::= x = y \mid R(z) \mid \varphi \lor \varphi \mid \neg \varphi$$

**Definition 1** (Program Verification Modulo Axioms). For a program $s$ and a set of axioms $\mathcal{A}$, we say that $s$ satisfies a postcondition $\varphi$ over the syntax $\mathcal{L}$ modulo $\mathcal{A}$ if for every $\mathcal{A}$-model $M$ and for execution $\rho \in \text{Exec}(s)$ that is feasible in $M$, $M$ satisfies $\varphi[\text{eval}_M(\rho, V)/V]$ (i.e., where each variable $x \in V$ is replaced by $\text{eval}_M(\rho, V)$).

We remark that one can alternatively phrase the verification problem stated above in terms of feasibility. That is, a program $s$ satisfies a postcondition $\varphi$ modulo $\mathcal{A}$ iff every execution $\rho$ of $s'$ is infeasible modulo $\mathcal{A}$ (i.e., there is no $\mathcal{A}$-model $M$ such that $\rho$ is feasible in $M$), where $s' = s; \text{assume}(\neg \varphi)$.

4 COHERENCE MODULO AXIOMS

In this section we extend the notion of coherence from [Mathur et al. 2019a], adapting it to our current setting where we restrict data models using axioms $\mathcal{A}$. We will first recall the notion of terms computed by an execution, which will be used to define the notion of coherence.

4.1 Terms Computed and Assumptions Accumulated by Executions

We will associate a syntactic term with each variable after a partial execution $\rho$. This, intuitively, is the term computed by $\rho$ and stored in $x$. Let $\text{Terms}_\Sigma$ be the set of terms built using constants and functions in $\Sigma$. The term stored in $x$ after $\rho$ is defined inductively on $\rho$ as follows.

\[
\begin{align*}
\text{TEval}(\epsilon, x) &= \tilde{x} & \text{for every } x \in V \\
\text{TEval}(\rho \cdot "x := y", z) &= \text{TEval}(\rho, y) & \text{if } z \text{ is } x \\
\text{TEval}(\rho \cdot "x := f(z_1, \ldots, z_r)", y) &= f(\text{TEval}(\rho, z_1), \ldots, \text{TEval}(\rho, z_r)) & \text{if } y \text{ is } x \\
\text{TEval}(\rho \cdot a, x) &= \text{TEval}(\rho, x) & \text{otherwise}
\end{align*}
\]

The set of terms computed by an execution $\rho$ is $\text{Terms}(\rho) = \{ \text{TEval}(\rho', x) \mid \rho' \text{ is a prefix of } \rho, x \in V \}$. As an execution proceeds, it accumulates assumptions over the terms it computes, and we will use $\kappa(\rho)$ to denote the assumptions made by the execution $\rho$. In [Mathur et al. 2019a], relations are modeled using functions (to Booleans) and hence relational assumes were avoided. In the current exposition, however, we will treat relations as first class objects and the set of assumptions will also include relational predicates. Formally, $\kappa(\rho)$ is a set of ground predicates over $\Sigma \cup \{=\}$ defined...
as follows.
\[
\begin{align*}
\kappa(\epsilon) &= \emptyset \\
\kappa(\sigma \cdot \text{"assume}(x = y)\}) &= \kappa(\sigma) \cup \{\text{TEval}(\sigma, x) = \text{TEval}(\sigma, y)\} \\
\kappa(\sigma \cdot \text{"assume}(x \neq y)\}) &= \kappa(\sigma) \cup \{\text{TEval}(\sigma, x) \neq \text{TEval}(\sigma, y)\} \\
\kappa(\sigma \cdot \text{"assume}(R(z_1, z_2, \ldots, z_k))\}) &= \kappa(\sigma) \cup \{R(\text{TEval}(\sigma, z_1), \ldots, \text{TEval}(\sigma, z_k))\} \\
\kappa(\sigma \cdot \text{"assume}(\neg R(z_1, z_2, \ldots, z_k))\}) &= \kappa(\sigma) \cup \{\neg R(\text{TEval}(\sigma, z_1), \ldots, \text{TEval}(\sigma, z_k))\} \\
\kappa(\sigma \cdot a) &= \kappa(\sigma) \quad \text{otherwise}
\end{align*}
\]

### 4.2 Coherence

Our definition of coherence modulo axioms is a smooth generalization of the definition of coherence in [Mathur et al. 2019a]. The notion of coherence consists of two properties — memoizing and early equality assumes. The memoizing property says, intuitively, when a term \(t\) is computed after executing some prefix \(\sigma\) of an execution, if \(t\) is equivalent to some other term modulo the assumptions made in the execution so far, then \(t\) must not have been dropped at the end of \(\sigma\), i.e., a program variable must already hold this term. We replace the notion of equivalence of terms in this definition by equivalence modulo the axioms as well.

The notion of early assumes in [Mathur et al. 2019a] intuitively says that assumptions of equality (on terms \(t_1\) and \(t_2\)) should be encountered early — earlier than dropping any superterm of \(t_1\) or \(t_2\). This notion of early assumes allows for effectively computing congruence closure on the set of terms computed by the execution, which in turn, is necessary to accurately maintain which terms are equivalent. However, we observe that the notion in [Mathur et al. 2019a] is too restrictive and not entirely necessary. In our paper, we generalize this notion in several ways, to a more semantic one as follows. Whenever an execution encounters an assumption of equality between two term, we instead demand that only the equivalences that are additionally implied by this new assumption, can be inferred locally using the already known congruence between terms in the window, i.e., the set of terms pointed to by the program variables when the equality assumption is encountered. Next, we incorporate axioms into this definition, by requiring that the notion of equivalence is also modulo the axioms, and further require that all assumptions (equality, disequality, relational) are required to be early (as against only restricting equality assumptions to be early like in [Mathur et al. 2019a]). We will elaborate on these differences using an example after presenting the formal definition next.

Given a set of first order sentences \(\Gamma\) and ground terms \(t_1\) and \(t_2\), we say that \(t_1 \equiv_{\Gamma} t_2\) if \(\Gamma \models t_1 = t_2\).

**Definition 2 (Coherence modulo axioms).** Let \(\mathcal{A}\) be a set of axioms and let \(\rho\) be a complete or partial execution over variables \(V\). Then, \(\rho\) is said to be coherent modulo \(\mathcal{A}\) if it satisfies the following two properties.

**Memoizing.** Let \(\pi = \sigma \cdot "x := f(z)"\) be a prefix of \(\rho\) and let \(t = \text{TEval}(\pi, x)\). If there is a term \(t' \in \text{Terms}(\sigma)\) such that \(t' \equiv_{\mathcal{A} \cup \kappa(\sigma)} t\), then there must exist some variable \(y \in V\) such that \(\text{TEval}(\sigma, y) \equiv_{\mathcal{A} \cup \kappa(\sigma)} t\).

**Early Assumes.** Let \(\pi = \sigma \cdot "\text{assume}(c)"\) be a prefix of \(\rho\), where \(c\) is any of \(x = y, x \neq y, R(z)\), or \(\neg R(z)\). Let \(t \in \text{Terms}(\sigma)\) be a term computed in \(\sigma\) such that \(t\) has been dropped, i.e., for every \(x \in V\), we have \(\text{TEval}(\sigma, x) \not\equiv_{\mathcal{A} \cup \kappa(\sigma)} t\). For any term \(t' \in \text{Terms}(\sigma)\), if \(t \equiv_{\mathcal{A} \cup \kappa(\pi)} t'\), then \(t \equiv_{\mathcal{A} \cup \kappa(\sigma)} t'\).

**Remark.** We remark that every execution that is coherent as per the definition in [Mathur et al. 2019a], is also coherent modulo \(\mathcal{A} = \emptyset\) as in Definition 2. However, the converse is not true and we illustrate this difference below.
Example 3. Let us fix $\mathcal{A} = \emptyset$ for this example. Consider the execution $\rho = \sigma \cdot \text{assume}(x = y)$ where,
\[
\sigma = \text{assume}(x = y) \cdot x' := f(x) \cdot y' := f(y) \cdot x' := f(x') \cdot y' := f(y')
\]
We first observe that the prefix $\sigma$ is coherent both with respect to the definition in [Mathur et al. 2019a] and Definition 2. First there are no superterms of $\sigma = T\text{eval}(e, x)$ and $\tilde{y} = T\text{eval}(e, y)$ when the first statement $\text{assume}(x = y)$ is observed, and thus, this assume is early. Second, even though the statements “$y' := f(y)$” and “$y' := f(y')$” are computing a term that has been equivalently computed before (modulo the assumption $(\tilde{x} = \tilde{y})$), a copy of these terms is available in some program variable (variable $x'$ in both the cases) at the time of the execution, thus respecting the memoizing restriction.

Now let us discuss the execution $\rho$. This execution is not coherent with respect to [Mathur et al. 2019a]. In particular, the last assume $\text{assume}(x = y)$ is not early, as superterms $f(x)$ and $f(\tilde{y})$ have been computed but dropped in the prefix $\sigma$. However, observe that $f(\tilde{x}) \equiv_{\mathcal{A} \cup \kappa(\sigma)} f(\tilde{y})$ (here, $\mathcal{A} \cup \kappa(\sigma) = (\tilde{x} = \tilde{y})$) and thus, $\rho$ meets the early assumes restriction as per Definition 2, making $\rho$ coherent.

Let us now consider an example which illustrates the notion of coherence in the presence of axioms.

Example 4. Consider the execution $\rho$ below.
\[
\rho = z_1 := f(x, y) \cdot z_2 := f(y, x) \cdot z_3 := g(z_1) \cdot z_4 := g(z_2) \cdot z_5 := z_6 := g(z_1)
\]
Let $\rho_i$ denote the prefix of $\rho$ of length $i$. Here, $T\text{eval}(\rho_3, z_3) = g(f(\tilde{x}, \tilde{y}))$, $T\text{eval}(\rho_5, z_3) = \tilde{z}_5 \neq g(f(\tilde{x}, \tilde{y}))$ and $T\text{eval}(\rho_6, z_6) = g(f(\tilde{x}, \tilde{y}))$. When the set of axioms is $\mathcal{A} = \emptyset$, this execution is not coherent modulo $\mathcal{A}$ as it violates the memoizing requirement at the last statement $z_6 := g(z_1)$ (no variable stores the term $g(f(\tilde{x}, \tilde{y}))$ after $\rho_5$).

Now, consider the axiom set denoting commutativity of $f$, i.e., $\mathcal{A}_{\text{comm}} = \{ \forall u, v. f(u, v) = f(u, v) \}$. In this case, we observe that $f(\tilde{x}, \tilde{y}) \equiv_{\mathcal{A}_{\text{comm}}} f(\tilde{y}, \tilde{x})$ and thus $g(f(\tilde{x}, \tilde{y})) \equiv_{\mathcal{A}_{\text{comm}}} g(f(\tilde{y}, \tilde{x}))$. Also, $T\text{eval}(\rho_5, z_4) = g(f(\tilde{x}, \tilde{y})) \equiv_{\mathcal{A}_{\text{comm}}} g(f(\tilde{x}, \tilde{y}))$. This ensures that $\rho$ is indeed coherent modulo $\mathcal{A}_{\text{comm}}$.

Let $\text{CoherentExecs}(\Sigma, V, \mathcal{A})$ denote the set of executions over the signature $\Sigma$ and variables $V$ that are coherent modulo the set of axioms $\mathcal{A}$.

Definition 3. A program $s$ over signature $\Sigma$ and variables $V$ is said to be coherent modulo $\mathcal{A}$ if $\text{Exec}(s) \subseteq \text{CoherentExecs}(\Sigma, V, \mathcal{A})$.

In this paper, we explore several classes of axioms, studying when the verification problem for coherent programs modulo the axioms is decidable.

4.3 Recap of results from [Mathur et al. 2019a]

We briefly state the main decidability results from [Mathur et al. 2019a] about coherent programs, using the notation defined above, so the set of axioms $\mathcal{A}$ is empty. The results hold even when the early assumes condition is generalized (Definition 2) and relations are treated as first class objects, as we do in this paper.

**Theorem 1** (Essentially [Mathur et al. 2019a]). Let $\Sigma$ be a first order signature and $V$ a finite set of variables. The following observations hold when the set of axioms is empty:

1. There is a finite automaton $F$ (effectively constructable) of size $O(2^{\text{poly}(|V|)})$ such that for any coherent execution $\rho$, $F$ accepts $\rho$ iff $\rho$ is feasible.
(2) There is a finite automaton $C$ (effectively constructible) of size $O(2^{\text{poly}(|V|)})$ such that $L(C) = \text{CoherentExecs}(\Sigma, V, \emptyset)$.

As a consequence, the following problems are decidable in PSPACE.

- Given a coherent program $P$, determine if $P$ is correct.
- Given a program $P$, determine if $P$ is coherent.

The problems of verifying coherent programs and checking coherence, are also PSPACE-hard.

**Proof Sketch.** These observations have been proved in [Mathur et al. 2019a], but the proof is also sketched in Appendix A for completeness and to account for the modified definitions.

Intuitively, the automata to check feasibility and coherence of executions, track equivalences between program variables, functional and relational correspondences between them that hold based on the assumes seen. Crucial to establishing the correctness of the automata constructions is the observation that, when the set of axioms is empty, equality of two terms does not depend on disequality and relational assumes seen in the execution. That is, if $\kappa(\rho)_{eq}$ denotes the set of equality assumes in $\rho$, then for any computed terms $t_1, t_2$, $t_1 \equiv_{\kappa(\rho)} t_2$ iff $t_1 \equiv_{\kappa(\rho)_{eq}} t_2$.  

\[ \square \]

5 AXIOMS OVER RELATIONS

In this section, we investigate the decidability of the verification problem for coherent programs modulo relational axioms, i.e., axioms which only involve relation symbols $R$ in the signature $\Sigma$.

5.1 Verification modulo EPR axioms

A first-order formula is said to be an EPR formula [Ramsey 1987] if it is of the form

$$\exists x_1 \ldots x_k \forall y_1, \ldots, y_m \varphi$$

where $\varphi$ is quantifier-free and purely relational (does not use any function symbols).

It is well known that satisfiability of EPR formulas is decidable, in fact by a reduction to Boolean satisfiability [Lewis 1980]. Consequently, the problem of checking whether a single execution is feasible under axioms written in EPR can be shown to be decidable, and has been exploited in bounded model-checking.

Consequently, we could reasonably ask whether verification of coherent programs under EPR axioms is decidable. Surprisingly, we show that they are not.

**Theorem 2.** Verification of uninterpreted coherent programs modulo EPR axioms is undecidable.  

**Proof.** The undecidability is proved through a reduction from Post’s Correspondence Problem (PCP). Recall that PCP is the following problem.

**PCP.** Let $\Delta = \{a_1, a_2, \ldots, a_k\}$ be a finite alphabet ($|\Delta| > 2$). Given strings $\alpha_1, \alpha_2, \ldots, \alpha_N, \beta_1, \beta_2, \ldots, \beta_N \in \Delta^*$ (with $N > 0$), determine if there is a sequence $i_1, i_2, \ldots, i_M$ such that $1 \leq i_j \leq N$ for every $1 \leq j \leq M$ and

$$\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_M} = \beta_{i_1} \cdot \beta_{i_2} \cdots \beta_{i_M}$$

It is well known that the PCP problem is undecidable. We will prove that given a PCP instance $I = (\Delta, \alpha_1, \ldots, \alpha_N, \beta_1, \ldots, \beta_N)$, we can construct a set of EPR axioms $\mathcal{A}$, a program $P_{EPR}$ that is coherent with respect to $\mathcal{A}$, and a postcondition $\phi$ such that $I$ is a YES instance of PCP iff $P_{EPR}$ does not satisfy $\phi$.

Let us fix a PCP instance $I$. The desired program $P_{EPR}$ (with post condition $\phi$) is shown in Fig. 3 and the set of EPR axioms $\mathcal{A}$ is shown in Fig. 4.

The signature $\Sigma$ consists of unary functions $f, g$ and $s$. The set of relations in $\Sigma$ is

$$\mathcal{R} = \{ R, S \} \cup \{ Q_a \mid a \in \Delta \}.$$
(* For 1 ≤ p ≠ q ≤ N−1 *)

assume (z_p ≠ z_q);

assume (R(x, y));
assume (z_0 ≠ M);
j := z_0;
while (j ≠ M) {

i_j := g(j);

if (i_j = z_1) {

x' := f(x);
assume (S(x, x'));
assume (Q_{α_{i_1}}(x'));
x := x';
:
}
x' := f(x);
assume (S(x, x'));
assume (Q_{α_{i_1}}(x'));
x := x';
y' := f(y);
assume (S(y, y'));
assume (Q_{β_{i_1}}(y'));
y := y';
:
y' := f(y);
assume (S(y, y'));
assume (Q_{β_{i_1}}(y'));
y := y';
}
j := s(j);
@post: ¬R(x, y)

Fig. 3. Program $P_{EPR}$ for showing verification is undecidable when there are relations and obey EPR axioms in Fig. 4.

The relations $R$ and $S$ are binary, while the rest are unary relations. The set of variables in the program are

$\mathcal{V} = \{z_1, \ldots, z_{N-1}\} \cup \{x, x', y, y', z_0, j, i_j, M\}$

Intuitively, the program constructs two strings that prove that $I$ is a YES instance of PCP — the positions on one string are indexed by the variable $x$ and positions on the second string are indexed by the variable $y$. Variable $M$ intuitively stores the number of $α_i$'s that need to be concatenated to get a solution. The value of $M$ is fixed by the data model; this way of exploiting data models to get "nondeterminism" is key in this reduction. The variable $z_0$ stores "0", and the variables $z_i$ ($i > 0$) store indices of strings in the input instance $I$. In each iteration of the while-loop, the index of the next pair of strings in the solution is "picked" by applying the (uninterpreted) function $g$; here again the data model that interprets $g$ resolves the non-determinism. Once the index is picked, the appropriate strings are "concatenated". This happens step-by-step by generating the next index.
\begin{equation}
\forall x, y_1, y_2 \cdot R(x, y_1) \land R(x, y_2) \implies y_1 = y_2
\end{equation}
(1)
\begin{equation}
\forall x_1, x_2, y \cdot R(x_1, y) \land R(x_2, y) \implies x_1 = x_2
\end{equation}
(2)
\begin{equation}
\forall x, y_1, y_2 \cdot S(x, y_1) \land S(x, y_2) \implies y_1 = y_2
\end{equation}
(3)
\begin{equation}
\forall x_1, y_1, x_2, y_2 \cdot R(x_1, y_1) \land S(x_1, x_2) \land S(y_1, y_2) \implies R(x_2, y_2)
\end{equation}
(4)
For every \( a \in \Delta \), we have \( \forall x, y : R(x, y) \land Q_a(x) \implies Q_a(y) \)
(5)
For every \( a \in \Delta \), we have \( \forall x, y : R(x, y) \land Q_a(y) \implies Q_a(x) \)
(6)
For every \( a \neq b \in \Delta \), we have \( \forall x, y : Q_a(x) \implies \neg Q_b(x) \)
(7)

Fig. 4. Axioms for the relations used in \( P_{EPR} \).

by applying function \( f \), and fixing the symbol at that position. Here the relation \( Q_a \) plays a role; if \( Q_a(x) \) holds then intuitively it means that symbol \( a \) appears in position \( x \) of the string. Finally, after the next pair is concatenated, the index of the number of strings in the solution (a.k.a. \( j \)) is “incremented” (by using \( s \)).

The relations \( R \) and \( S \) play an important role. \( S \) is the successor relation on string positions, and so appropriate assumptions on \( S \) are inserted whenever \( f \) is used. The relation \( R \) relates positions of the two constructed strings if the prefix up to that position is identical in the two strings — we start with requiring that the first positions are related by \( R \) and our post condition demands that the last two positions are not \( R \)-related to say that the constructed strings are not a solution to the PCP instance.

The axioms in \( A \) ensure that the relations \( R, S, \) and \( Q_a \) are interpreted consistently with the above intuition. Axioms (1) and (2) require that a position in the first/second string is \( R \) related to at most one position in the second/first string. Axioms (5) and (6) say that \( R \)-related positions have the same symbol, while axiom (4) says that if two positions are \( R \) related then so are their “successors” (i.e., \( S \)-related elements). Axiom (3) requires \( S \) to behave like a successor relation — any position as at most one \( S \)-related position. Finally axiom (7) intuitively says that there is at most one symbol at any position.

The correctness of this reduction and the fact that \( P_{EPR} \) is coherent modulo \( A \) is proved in Appendix B.1; it is based on the intuitions outlined above. \( \square \)

Given the above result, we turn to several classes of quantified axioms, which are all expressible in EPR (and hence have a decidable bounded model checking problem) and examine their decidability for coherent program verification.

### 5.2 Reflexivity, Irreflexivity, and Symmetry

We now examine coherent program verification under the following axioms (individually):

\begin{align*}
\varphi^R_{\refl} &= \forall x : R(x, x) \quad \text{(reflexivity)}
\varphi^R_{\irref} &= \forall x : \neg R(x, x) \quad \text{(irreflexivity)} \\
\varphi^R_{\symm} &= \forall x, y : R(x, y) \implies R(y, x) \quad \text{(symmetry)}
\end{align*}

We show that verification is decidable modulo these axioms using a technique that we call program instrumentation. Let us fix a relation \( R \) and an axiom \( \varphi^R_P \), where \( p \in \{ \refl, \irref, \symm \} \). The idea is to find a function (in fact, a string homomorphism) \( h \) such that for any program \( P \), \( P \) is correct/coherent modulo \( \varphi^R_P \) iff \( h(\text{Exec}(P)) \) is correct/coherent modulo the empty axiom set. Decidability then follows by exploiting the results of thmrefpopl19. The function \( h \) will capture the
properties of the axiom it is trying to eliminate, and so it will be different for different axioms. We first outline these function \( h \), then state their property and prove the decidability result.

For reflexivity, we transform an execution \( \rho \) of \( P \) to \( \rho' \) where \( \rho' \) is essentially \( \rho \), except that whenever we see the computation of a term, using an assignment of the form \( \langle x := f(z) \rangle \), we immediately insert an assume statement that states that \( R(x, x) \) holds. More precisely, the homomorphism is defined as,

\[
h^R_{\text{refl}}(a) = \begin{cases} a \cdot \text{"assume}(R(x, x))" & \text{if } a = \langle x := f(z) \rangle \\ a & \text{otherwise} \end{cases}
\]

For irreflexivity, whenever we see the computation of a term using an assignment of the form \( \langle x := f(z) \rangle \), we immediately insert an assume statement that demands that \( \neg R(x, x) \) holds. That is, we instrument executions using the following homomorphism.

\[
h^R_{\text{irref}}(a) = \begin{cases} a \cdot \text{"assume}(\neg R(x, x))" & \text{if } a = \langle x := f(z) \rangle \\ a & \text{otherwise} \end{cases}
\]

Finally, for the symmetry axiom on a relation \( R \), whenever we see an assumption of the form \( \text{"assume}(R(x, y))" \), we insert an assumption that \( R(y, x) \) holds. In other words, we use the following homomorphism.

\[
h^R_{\text{symm}}(a) = \begin{cases} a \cdot \text{"assume}(R(y, x))" & \text{if } a = \text{"assume}(R(x, y))" \\ a \cdot \text{"assume}(\neg R(y, x))" & \text{if } a = \text{"assume}(\neg R(y, x))" \\ a & \text{otherwise} \end{cases}
\]

The key property of these homomorphisms is that they preserve coherence and feasibility of executions. We state and prove a slightly general form of this statement next.

**Lemma 3.** Let \( \mathcal{A} \) be a set of first order sentences such that the relation symbol \( R \) does not syntactically appear in any sentence in \( \mathcal{A} \). Let \( \rho \in \{ \text{refl, irref, symm} \} \). For any execution \( \rho \) the following two properties hold.

1. \( \rho \) is feasible modulo \( \mathcal{A} \cup \{ \phi^R_\rho \} \) iff \( h^R_\rho(\rho) \) is feasible modulo \( \mathcal{A} \). Here \( \rho \) may or may not be coherent.

2. \( \rho \) is coherent modulo \( \mathcal{A} \cup \{ \phi^R_\rho \} \) iff \( h^R_\rho(\rho) \) is coherent modulo \( \mathcal{A} \).

**Proof Sketch.** No matter what \( p \) is, the proof follows roughly the same broad outline. First we show that for any execution \( \rho \) (which may or may not be coherent), \( \rho \) is feasible modulo \( \mathcal{A} \cup \{ \phi^R_\rho \} \) iff \( h^R_\rho(\rho) \) is feasible modulo \( \mathcal{A} \). Observe that one direction of this is straightforward. If \( \rho \) is feasible in a \( \mathcal{A} \cup \{ \phi^R_\rho \} \)-model \( M \) then based on how \( h^R_\rho \) is defined, \( h^R_\rho(\rho) \) is also feasible in \( M \). In addition by definition, \( M \) is a \( \mathcal{A} \)-model, which completes this direction. For the converse, suppose \( h^R_\rho(\rho) \) is feasible in a \( \mathcal{A} \)-model \( M \). To prove that \( \rho \) is feasible in a \( \mathcal{A} \cup \{ \phi^R_\rho \} \) model, we look at a model \( M' \) which agrees with \( M \) in the interpretation of all symbols except \( R \). For \( R \), we take the “smallest” set of pairs that need to hold based on the \textit{assumes} in \( h^R_\rho(\rho) \) and the axiom \( \phi^R_\rho \); the definition of \( M' \) depends on \( \rho \). The exact construction of \( M' \) and its correctness is described fully in Appendix B.2.

To establish coherence preservation, a crucial observation turns out to be that equivalences on terms modulo \( \mathcal{A} \cup \{ \phi^R_\rho \} \) in \( \rho \) is the same as equivalences on terms modulo \( \mathcal{A} \) in \( h(\rho) \). More precisely, for any execution \( \rho \), and any two computed terms \( t_1, t_2 \in \text{Terms}(h^R_\rho(\rho)) \),

\[
t_1 \equiv_{\mathcal{A} \cup \{ \phi^R_\rho \} \cup \mathcal{K}(\rho)} t_2 \iff t_1 \equiv_{\mathcal{A} \cup \mathcal{K}(h(\rho))} t_2.
\]

Using this property, one can argue the preservation of coherence. Again details can be found in Appendix B.2. □
We now consider the transitivity axiom for a relation $R$. This is because transitivity effects can be global. For example, we may have

$$R \exists xy \neg R \exists yz \implies \exists xz \neg R$$

This can be implied from the relations marked by bold edges ($\Rightarrow$).

**Theorem 4.** For any relation symbol $R$ and $p \in \{\text{refl}, \text{irref}, \text{symm}\}$, the problems of coherent verification modulo $\{\varphi_p^R\}$ and checking coherence modulo $\{\varphi_p^R\}$ are PSPACE-complete.

**Proof.** The result is essentially a corollary of Lemma 3 and Theorem 1. Since $h_p^R$ is a homomorphism, $h_p^R(\text{Exec}(P))$ for any program $P$ is regular. By Lemma 3, $P$ is correct/coherent modulo $\{\varphi_p^R\}$ iff $h_p^R(\text{Exec}(P))$ is correct/coherent modulo $\emptyset$. The upper bound follows by exploiting the automata constructed in the proof Theorem 1. \qed

### 5.3 Transitivity

We now consider the transitivity axiom for a relation $R$ which says

$$\varphi_{\text{trans}}^R = \forall x, y, z : R(x, y) \land R(y, z) \implies R(x, z) \quad \text{(transitivity)} \quad (9)$$

We prove that coherent program verification and checking coherence modulo transitivity axioms is decidable. However, the proof is different and more complex that the proofs for reflexivity, irreflexivity, and symmetry. Intuitively, the program instrumentation approach does not seem to work for transitivity. This is because transitivity effects can be global. For example, we may have that the execution asserts the sequence of relational assumes $R(t_1, t_2), R(t_2, t_3), \ldots R(t_{n-1}, t_n)$ (here, $t_1, \ldots t_n$ are terms computed by the execution), where some of the intermediate terms may have been dropped by the program (i.e., the variables holding these terms were reassigned). Consequently, relating $t_1$ and (the possibly newly constructed term) $t_n$ requires that the automaton keeps the transitive closure of the assumptions the program makes. Assumptions of negations of relations done by the program also need special care.

Our main observation about transitivity is the following.

**Theorem 5.** Let $\Sigma$ be a first order signature and $V$ a finite set of program variables. Let $\mathcal{A} = \{\varphi_{\text{trans}}^R | R \in \mathcal{R}_{\text{trans}}\}$ for some set of relation symbol $\mathcal{R}_{\text{trans}}$ in $\Sigma$. The following observation hold.

1. There is a finite automaton $F_{\text{trans}}$ (effectively constructable) of size $O(2^{\text{poly}(|V|)})$ such that for any coherent execution $\rho$, $F$ accepts $\rho$ iff $\rho$ is feasible.

2. There is a finite automaton $C_{\text{trans}}$ (effectively constructable) of size $O(2^{\text{poly}(|V|)})$ such that $L(C) = \text{CoherentExecs}(\Sigma, V, \mathcal{A})$.

**Proof Sketch.** These are in some sense a generalization of the automata constructions used to establish Theorem 1. Like before, the automata $F_{\text{trans}}$ and $C_{\text{trans}}$ rely on tracking equivalence between values stored in variables, and functional and relational correspondences between these values. However, now since some relations maybe transitive, additional relational correspondences (or their absence) maybe implied for $R \in \mathcal{R}_{\text{trans}}$. The basic idea is to maintain for transitive relations $R$ (a) the transitive closure of the positive relation assumes $\text{assume}(R(\cdot, \cdot))$, and (b) the negative relational assumes implied by the relational assumes seen in an execution. More precisely, if the execution sees assumes $\text{assume}(R(x, y))$ and $\text{assume}(R(y, z))$, then we also add the constraint $R(x, z)$ in the
automaton’s state. Further, if the execution observes \textbf{assume}(R(x, y)) and \textbf{assume}(\neg R(x, z)) \textbf{PSPACE-complete}, then one can infer the constraint \neg R(y, z), and in this case, we accumulate this additional constraint in the state of the automaton. Similarly, if the execution observes \textbf{assume}(R(y, z)) and \textbf{assume}(\neg R(x, z)) \textbf{PSPACE-complete}, then one can infer the constraint \neg R(x, y), which is added in the automaton’s state. Both these scenarios are illustrated in Fig. 5. A detailed proof of this result is given in Appendix B.3.

Like in the proof of Theorem 1, a crucial property to establish correctness is that for transitivity axioms, the equality of two terms does not depend on the disequality and relational assumes seen in the execution. Establishing this property relies on showing that if an execution is feasible in a model then there is a canonical, \textit{minimal} model in which it is feasible. Existence of canonical models maybe of independent interest. Full details can be found in Appendix B.3. □

As a consequence we have the following result.

\textbf{Theorem 6.} For \(\mathcal{A} = \{\varphi_{\text{trans}}^R \mid R \in \mathcal{R}_{\text{trans}}\}\), the problems of coherent verification modulo \(\mathcal{A}\) and checking coherence modulo \(\mathcal{A}\) are \textbf{PSPACE-complete}.

\textbf{Proof.} Based on Theorem 5 and the fact that the set of executions of a program is regular, we reduce these problems to checking the emptiness of a regular language. The size of the resulting automata is exponential in \(|V|\), and so the \textbf{PSPACE} upper bound follows. □

5.4 Strict Partial Orders
We now turn our attention to axioms that dictate that certain relations be partial or total orders. The anti-symmetry axiom that holds for non-strict orders introduces subtle complications. Recall that \(R\) is anti-symmetric if \(\forall x, y. R(x, y) \land R(y, x) \Rightarrow x = y\); this axiom can imply equality between terms if \(R\) holds between a pair of terms. Concretely, if \(R\) is anti-symmetric, and the program makes assumptions in an execution that \(R(t_1, t_2)\) and \(R(t_2, t_1)\) hold, then any model in which such an execution is feasible must also ensure that \(t_1 = t_2\). This implicit equality assumption interferes with the notions of coherence and the automata constructions (proofs of Theorems 1 and 5) that compute a congruence closure on terms in a streaming fashion. Notice crucial to the correctness of automata construction used in the proofs of Theorems 1 and 5 was the observation that disequality and relational assumes could be disregarded to determine if two terms are equal. This would break in the presence of an anti-symmetric relation.

Hence, we only consider \textit{strict} partial orders in this section. Recall that a relation \(R\) is a strict partial order if it satisfies the irreflexivity axiom and the transitivity axiom, together denoted \(\mathcal{A}^R_{\text{SPO}}\). We can prove decidability for problems modulo \(\mathcal{A}^R_{\text{SPO}}\) by using our algorithm for irreflexivity and transitivity.

\textbf{Theorem 7.} The following problems are \textbf{PSPACE-complete}.

\begin{enumerate}
  \item Given a program \(P\) that is coherent modulo \(\mathcal{A}^R_{\text{SPO}}\), determine if \(P\) is correct.
  \item Given a program \(P\), determine if \(P\) is coherent modulo \(\mathcal{A}^R_{\text{SPO}}\)
\end{enumerate}

\textbf{Proof Sketch.} The proof is similar to the proof of Theorem 4. The idea is to show that an execution \(\rho\) is coherent/feasible modulo \(\mathcal{A}^R_{\text{SPO}}\) iff \(h^R_{\text{irref}}(\rho)\) is coherent/feasible modulo \(\varphi_{\text{trans}}^R\). This observation does not follow from Lemma 3 and it requires that lemma to be generalize. We will do that and actually prove a more general result about combination of axioms (Theorem 16), and the current theorem will be a corollary of that observation. □

5.5 Strict Total Orders
A relation \(R\) is a strict total order if it is a strict partial order and satisfies the \textit{totality axiom}:

\[ \forall x, y \cdot x \neq y \Rightarrow R(x, y) \lor R(y, x) \quad \text{\textit{(totality)}} \tag{10} \]
Let \( \mathcal{A}^R_{STO} \) denote the axioms of irreflexivity, transitivity, and totality for the relation \( R \).

Strict total orders are again tricky to handle as the axiom for totality can result in implicit equality between terms. For example, if \( \neg R(x, y) \) and \( \neg R(y, x) \) then it must be the case that \( x = y \). However, if we restrict ourselves to executions that only have assumes of the form \( \text{assume}(R(x, y)) \) and do not have any assumes on \( \neg R \), i.e., of the form \( \text{assume}(\neg R(x, y)) \) then there are no implicit equalities that are entailed.

Unfortunately, in general, program executions can contain negative assumes on \( R \) (i.e., assumes of the form \( \text{assume}(\neg R(x, y)) \)). In order to ensure that executions contain only positive assumptions on \( R \), we must be careful when identifying executions of programs with conditionals — branches where the assumption \( \neg R(x, y) \) holds must be translated to a branch that assumes \( R(y, x) \) and a branch that assumes \( x = y \). That is, we modify the following rules defining executions of programs for branch statements; for all other statements, the rules are the same as in Section 3.2.

\[
\begin{align*}
\text{Exec( } \text{assume}(\neg R(x, y)) \text{ )} &= \text{“assume}(R(y, x)) \text{” + “assume}(x = y)\text{”} \\
\text{Exec( if } R(x, y) \text{ then } s_1 \text{ else } s_2 \text{ )} &= \text{“assume}(R(x, y))\text{”} \cdot \text{Exec}(s_1) + \text{Exec(assume}(\neg R(x, y))\text{)} \cdot \text{Exec}(s_2) \\
\text{Exec( if } \neg R(x, y) \text{ then } s_1 \text{ else } s_2 \text{ )} &= \text{Exec(assume}(\neg R(x, y))\text{)} \cdot \text{Exec}(s_1) + \text{“assume}(R(x, y))\text{”} \cdot \text{Exec}(s_2) \\
\text{Exec( while } R(x, y) \{ s \} \text{ )} &= \left[ \text{“assume}(R(x, y))\text{”} \cdot \text{Exec}(s_1) \right]^* \cdot \text{Exec(assume}(\neg R(x, y))\text{)} \\
\text{Exec( while } \neg R(x, y) \{ s \} \text{ )} &= \left[ \text{Exec(assume}(\neg R(x, y))\text{)} \cdot \text{Exec}(s_1) \right]^* \cdot \text{“assume}(R(x, y))\text{”}
\end{align*}
\]

Notice that executions can now have additional equality assumes even if they did not appear in the program. When refer to coherent programs, we mean that they are coherent according to the above modified notion of executions. This means for such programs to be coherent, all executions must ensure that the additional equality assumes are early. And when we talk about coherent verification of programs with total orders, we mean verification for programs that are coherent after this transformation.

We observe that when executions have only positive \( R \) assumes, checking properties modulo \( \mathcal{A}^R_{STO} \) is equivalent to checking properties modulo \( \mathcal{A}^R_{SPO} \). This will allow us to reduce the case of strict total orders to the case of strict partial orders.

**Lemma 8.** Let \( A \) be a set of first order sentences that do not mention \( R \). Let \( \rho \) be an execution that does not have any symbols of the form \( \text{“assume}(\neg R(x, y))\text{”} \). Then the following two observations hold.

1. \( \rho \) is feasible modulo \( A \cup \mathcal{A}^R_{STO} \) iff \( \rho \) is feasible modulo \( A \cup \mathcal{A}^R_{SPO} \); note that \( \rho \) may or may not be coherent.
2. \( \rho \) is coherent modulo \( A \cup \mathcal{A}^R_{STO} \) iff \( \rho \) is feasible modulo \( A \cup \mathcal{A}^R_{SPO} \).

**Proof.** Let us prove these properties in order. Observe that if \( \rho \) is feasible in a \( A \cup \mathcal{A}^R_{STO} \)-model \( M \) then since \( M \) is also a \( A \cup \mathcal{A}^R_{SPO} \)-model, we have \( \rho \) is feasible modulo \( A \cup \mathcal{A}^R_{SPO} \). Suppose \( \rho \) is feasible in a \( A \cup \mathcal{A}^R_{SPO} \)-model \( M \). Let \( M' \) have the same universe and interpretation for all symbols except \( R \). Let \( S \) be any linearization of \( [R]_M \) and take \( [R]_{M'} \) to be \( S \). Since \( R \) is not mentioned in any sentence in \( A \) and \( M \) and \( M' \) agree on all symbols except \( R \), \( M' \) is a \( A \)-model. Further by defininition, \( M' \) satisfies \( \mathcal{A}^R_{STO} \). Since \( [R]_M \subseteq [R]_{M'} \), \( M \) and \( M' \) agree on all symbols except \( R \), and \( \rho \) does not have any negative \( R \)-assumes, all assumes in \( \rho \) must hold in \( M' \) because they hold in \( M \).

The extension to prove coherence follows the proof template as the one to prove Lemma 3, and exploits the model construction outlined for feasibility above. The detailed proof is left to the reader. \( \square \)

**Theorem 9.** The problems of coherent verification, and checking coherence modulo \( \mathcal{A}^R_{STO} \) are PSPACE-complete.
Proof. Based on Lemma 8, we can reduce checking these properties modulo $\mathcal{R}_{SPO}$. Then the result follows from Theorem 7.

6 AXIOMS OVER FUNCTIONS

We now discuss computational problems modulo axioms that involve function symbols. The treatment of axioms involving functions in the verification of coherent programs is inherently hard. This is because, like in the case of (nonstrict) partial orders and strict total orders, the axioms along with the assume-steps in the execution, can imply equalities between terms beyond those entailed by just the assume steps in the execution. For example, consider the axiom $\forall x, y \cdot f(x, y) = f(y, x)$ constraining $f$ to be a commutative function. Then terms like $f(f(x, y), z)$ are equal to terms like $f(z, f(y, z))$, and hence when building models we must make sure that functions/relations on such terms are defined in the same way. Terms made equivalent by the functional axioms can be syntactically very different, and keeping track of the equivalence on unbounded executions is hard using finite memory. We consider many natural classes of axioms, and proving both positive and negative results that help delineate the decidability/undecidability boundary.

6.1 Associativity

We now consider the associativity axiom for a function $f$.

$$\phi_{assoc}^f = \forall x, y, z \cdot f(x, f(y, z)) = f(f(x, y), z) \quad \text{(associativity)} \quad (11)$$

We show, surprisingly to us, that coherent verification is undecidable modulo $\{\phi_{assoc}^f\}$, i.e., even when we have only one axiom that requires only one function to be associative. In fact, the situation is a lot worse — checking the feasibility of even a single (even coherent) execution is undecidable, in the presence of a single associative function.

To prove this result, we recall a classical computational problem called the word problem for a semi-group. Recall that a semi-group is an algebra consisting of a universe on which a single associative binary operation (often denoted $\circ$) is defined. A semi-group $S$ is generated from a finite set $\Delta$, every element in the universe of $S$ can be constructed starting from $\Delta$ using the operation $\circ$.

**Word Problem over Semi-Groups.** Let $\Delta$ be a finite set and $\circ$ be the concatenation operation. Given word identities $u_1 = v_1, u_2 = v_2, \ldots, u_n = v_n$, and an additional identity $u_0 = v_0$, determine if for any semi-group $S$ generated from $\Delta$ in which the identities $u_i = v_i$, for $1 \leq i \leq n$ hold, whether $u_0 = v_0$ holds.

This problem is known to be undecidable.

**Theorem 10 (Post’47 [Post 1947]).** Word problem for finitely generated semigroups is undecidable.

Using Post’s result, we prove undecidability to check the feasibility of a single coherent execution.

**Theorem 11.** Given a a trace $\rho$ that is coherent modulo $\{\phi_{assoc}^f\}$, it is undecidable to determine if $\rho$ is feasible. Therefore, the problem checking if a program $P$ that is coherent modulo $\{\phi_{assoc}^f\}$ is undecidable.

Proof. We show the following reduction. Given an instance $I = (\Delta, \circ, u_1, v_1, \ldots, u_n, v_n, u_0, v_0)$ there is an execution $\rho$ that is coherent modulo $\{\phi_{assoc}^f\}$ such that $I$ is a YES instance of the work problem iff $\rho$ is infeasible modulo $\{\phi_{assoc}^f\}$.
which obeys the associativity axiom $\phi$

The post-condition

$$\begin{array}{l}
\text{assume}(x_1 = y_1);
\end{array}$$

$\begin{array}{l}
\text{assume}(x_2 = y_2);
\end{array}$

... $\begin{array}{l}
\text{assume}(x_0 \neq y_0);
\end{array}$

Fig. 6. Execution $\rho_{assoc}$ for showing checking feasibility of a single coherent execution with one associative function is undecidable.

The constructed execution $\rho$ is shown in Fig. 6. The signature $\Sigma$ consists of a binary function $f$ which obeys the associativity axiom $\phi_{assoc}$. The set of variables in the program are

$$\mathcal{V} = \{a_1 \ldots a_k\} \cup \bigcup_{i=0}^{N} \{x_i, x_{i,1}, x_{i,2}, \ldots x_{i,|u_i|}\} \cup \bigcup_{i=1}^{N} \{y_i, y_{i,1}, y_{i,2}, \ldots y_{i,|v_i|}\}$$

The post-condition $\phi$ is $x_0 = y_0$.

Our reduction uses the associative function $f$ to model concatenation. A word $w = a_1, \ldots, a_m$ is modeled as the term $t_w = f(a_1, f(a_2, \ldots, f(a_{m-1}, a_m) \ldots))$. Intuitively, the execution first creates the words $u_1, v_1, u_2, \ldots, u_N, v_N$ and assumes $u_1 = v_1, u_2 = v_2, \ldots, u_N = v_N$. It then creates the words $u_0, v_0$ and checks $u_0 = v_0$ in the postcondition. Proof that $\rho$ is coherent and the reduction is correct is straightforward, but for completeness, the proof can be found in Appendix C.1.

\[\Box\]
6.2 Commutativity
We now consider the commutativity axiom, which is the following
\[ \varphi_{\text{comm}}^f = \forall x, y \cdot f(x, y) = f(y, x) \quad \text{(commutativity)} \]  
(12)
In this section we show that such axioms can be effectively handled.

We use the technique of program instrumentation to handle this axiom. We augment executions with an auxiliary variable \( v^* \notin V \) and further transform executions using the following homomorphism that uses the auxiliary variable \( v^* \)
\[ h_{\text{comm}}^f(a) = \begin{cases} a \cdot v^* := f(y, x) \cdot \text{assume}(z = v^*) & \text{if } a = z := f(x, y) \\
       a & \text{otherwise} \end{cases} \]

As in other cases where we used the program instrumentation technique (Section 5.2), the key to proving the correctness of this approach is to argue the homomorphism preserves feasibility and coherence.

Lemma 12. For any execution \( \rho \), the following properties hold.

1. \( \rho \) is feasible modulo \( \{ \varphi_{\text{comm}}^f \} \) iff \( h_{\text{comm}}^f(\rho) \) is feasible modulo \( \emptyset \).
2. \( \rho \) is coherent modulo \( \{ \varphi_{\text{comm}}^f \} \) iff \( h_{\text{comm}}^f(\rho) \) is coherent modulo \( \emptyset \).

Proof. As before, if \( \rho \) is feasible modulo \( \{ \varphi_{\text{comm}}^f \} \) then the same data model witnesses the fact that \( h_{\text{comm}}^f(\rho) \) is feasible modulo \( \emptyset \). Conversely, suppose \( h_{\text{comm}}^f(\rho) \) is feasible in a model \( M = (U_M, \sigma_M) \). Let \( e_1 \) be a new element not in \( U_M \). Define a new model \( M' \) with universe \( U_M \cup \{ e_1 \} \). The interpretation of relation symbols in \( M' \) is the same as that in \( M \). For \( k \)-ary function symbols \( g \), define
\[ \left[ g \right]_{M'}(e_1, \ldots, e_k) = \begin{cases} e & \text{if } e_1, \ldots, e_k \in S, \\
       & \text{and there are computed terms } t_1, \ldots, t_k \in \text{Terms}(\sigma) \\
       & \text{such that } t = g(t_1, \ldots, t_k) \in \text{Terms}(\sigma) \text{ and } e = \left[ t \right]_M \\
       e_{\text{fresh}} & \text{otherwise} \end{cases} \]

That is, we mimic the function interpretations on the set of computed terms (i.e., if a function interpretation was defined by the execution, we choose that), and point the functions to the new element \( e_{\text{fresh}} \) otherwise. We mimic the relational interpretations as exactly that of \( M \).

\[ \left[ f \right]_{M'} \] as defined is clearly commutative. This is because \( M \) satisfies the equality assumes in \( h_{\text{comm}}^f(\rho) \), which assert that for any computed term \( f(t_1, t_2) \), we have \( f(t_1, t_2) = f(t_2, t_1) \) holds. When either \( e_1, e_2 \) does not correspond to a computed term, we have \( \left[ f \right]_{M'}(e_1, e_2) = \left[ f \right]_{M'}(e_2, e_1) = e_{\text{fresh}} \). Further \( h_{\text{comm}}^f(\rho) \) is feasible in \( M' \). This is because on the computed set of terms, all equalities and disequalities are satisfied as they were also satisfied in \( M \). All relation assumes are only on the computed terms so they are also trivially satisfied.

This means that \( \rho \) is also feasible in the commutative model \( M' \) as the equalities of \( \rho \) are a subset of those in \( h_{\text{comm}}^f(\rho) \) and the disequalities and relational assumes of \( \rho \) are exactly that of \( h_{\text{comm}}^f(\rho) \).

Having established feasibility preservation, the proof of preservation of coherence follows the same outline as Lemma 3.

This gives us our next result:

Theorem 13. Verification of coherent programs and checking coherence modulo commutativity axioms is decidable and is \( \text{PSPACE}-\text{complete} \).
Proof. The proof uses Lemma 12 and is similar to Theorem 4.

6.3 Idempotence

Let us now consider the idempotence axiom. A unary function \( f \) is said to be idempotent if

\[
\phi^{f}_{\text{idem}} = \forall x \cdot f(x) = f(f(x))
\]

(idempotence) (13)

Here again, we use program instrumentation. We use an auxiliary variable \( v^* \) \( \notin V \) and use the following homomorphism:

\[
h^{f}_{\text{idem}}(a) = \begin{cases} 
    a \cdot "v^* := f(y)" \cdot \text{assume}(y = v^*) & \text{if } a = "y := f(x)" \\
    a & \text{otherwise}
\end{cases}
\]

Again we have a theorem about the preservation feasibility and coherence.

Lemma 14. For any execution \( \rho \), the following properties hold.

1. \( \rho \) is feasible modulo \( \{\phi^{f}_{\text{idem}}\} \) iff \( h^{f}_{\text{idem}}(\rho) \) is feasible modulo \( \emptyset \).
2. \( \rho \) is coherent modulo \( \{\phi^{f}_{\text{idem}}\} \) iff \( h^{f}_{\text{idem}}(\rho) \) is coherent modulo \( \emptyset \).

The proof of Lemma 14 is very similar to the proof of Lemma 12. The modified data model constructed is identical. The proof is therefore skipped. As before, the preservation theorem allows us to conclude the decidability of verification.

Theorem 15. Verification of coherent programs and checking coherence modulo idempotence axioms is PSPACE-complete.

7 COMBINING AXIOMS

We have thus far proved a number of decidability results when a relation or functions satisfies certain properties like reflexivity/irreflexivity/symmetry/transitivity or commutativity/idempotence. We now show that all of these results can be combined. That is, we can consider a signature where relations and functions are assumed to satisfy some subset of these properties, and with some being uninterpreted, and the verification problem will remain decidable for coherent programs.

Theorem 16. Let \( \mathcal{A} \) be a set of axioms where each relation symbol \( R \) is either a total order or satisfies some (possibly empty) subset of properties out of reflexivity, irreflexivity, symmetry, transitivity, and each function symbol \( f \) satisfies some (possibly empty) subset out of commutativity and idempotence. The verification problem for coherent programs modulo \( \mathcal{A} \) is PSPACE-complete.

A proof sketch of this theorem is provided in Appendix D. A simple consequence of Theorem 16 is that verification for coherent programs is decidable even when some of the relations are constrained to be equivalence relations or pre-orders.

8 RELATED WORK

The theory of equality with uninterpreted functions (EUF) is a widely used theory in many verification applications as it has decidable quantifier free fragment. EUF has been central to advances in verification of microprocessor control [Bryant et al. 2002; Burch and Dill 1994] and hardware verification [Andraus et al. 2008; Hojati et al. 1996] and property directed model checking [Ho et al. 2017]. EUF has been used as a popular abstraction in software verification [Babić and Hu 2007, 2008]. Uninterpreted functions have also been studied for equivalence checking and translation validation [Pnueli and Strichman 2006]. Recently, Bueno et al [Bueno and Sakallah 2019] demonstrated the effectiveness of uninterpreted programs for verifying SVCOMP benchmarks against control flow properties.
Mathur et al. [Mathur et al. 2019a] introduced the class of coherent uninterpreted programs and showed that verification of coherent programs, with or without recursive function calls, is a decidable problem. This is one of the few subclasses of program verification over infinite domains that is known to be decidable. Previous works [Godoy and Tiwari 2009; Gulwani and Tiwari 2007; Müller-Olm et al. 2005] have established decidability of verification of classes of uninterpreted programs with heavy syntactic restrictions such as disallowing conditionals inside loops or nested loops, etc. As noted in [Mathur et al. 2019a], the notion of coherence is close to the notion of a bounded pathwidth decomposition [Robertson and Seymour 1983]. A term that is created in a coherent execution stays within some program variable (modulo congruence) until the first time all variables containing that term are over-written, and after this point, the execution never computes it again, and thus, the set of windows that contain a term form a contiguous segment of the program execution. In general, path decomposition and the related notion of tree decomposition have been exploited many times in the literature to give decidability in verification [Chatterjee et al. 2016, 2015; Madhusudan and Parlato 2011].

The work in [Mathur et al. 2019b] exploits the notion of coherence to verify memory safety for a class of heap manipulating programs, including traversal algorithms on data structures such as singly linked list, sorted lists, binary search trees etc. While the reasoning for establishing memory safety can be often done by treating native data as purely uninterpreted, more complex reasoning would inadvertently be required for assertion checking, often requiring specific axioms to establish correctness. For example, a list search routine working over a sorted list (see Section 2) would require reasoning with the total order axioms.

The class of EPR formulas that consist of universally quantified formulas over relational signatures is a well-known decidable class of first-order logic [Ramsey 1987]. EPR-based reasoning has been proved powerful for verification of large-scale systems [McMillan 2016; Padon et al. 2017; Taube et al. 2018] and the Ivy [McMillan and Padon 2018; Padon et al. 2016b] system is one of the most notable framework that exploits EPR based reasoning for verifying program snippets without recursion. EPR encoding of order axioms such as reflexivity, symmetry, transitivity and total orders has been used in proving programs working over heaps [Itzhaky et al. 2014].

Another notable verification technique with an automata-theoretic foundations is the trace abstraction due to Heizmann et al [Farzan et al. 2013, 2014, 2015; Heizmann et al. 2009, 2010, 2013]. In this technique, one constructs a successively growing regular set that covers the executions of a program, and the correctness of a program can then eventually be asserted using emptiness question for regular languages.

9 CONCLUSIONS

By incorporating axioms on functions and relations, decidability results in this paper, enable a more faithfully automatic verification of programs. It is worth noting that the upper bound for all our decidability results is PSPACE, which is the same as that for Boolean programs. Thus, though we consider programs over infinite domains with additional structure, our verification results imply the same complexity as that for programs over finite domains.

One future direction is to adapt this technique for practical program verification. In this context, adapting our technique within the automata-theoretic technique of [Farzan et al. 2013, 2015; Heizmann et al. 2009, 2010, 2013] seems most promising. Second, there are several program verification techniques that use EPR, and in several of these, EPR is used mainly to establish a linear order on the universe [Itzhaky et al. 2014]. Automatically verifying such programs using our technique is worth exploring. On the theoretical front, there are several theories that are useful in program verification that could be explored for decidable coherent verification, such as updatable maps and arrays (where there are functions that can be updated point-wise, where the domain can be
arbitrary or have a linear order) and updatable sets. We believe that such theories would be useful in reasoning with heap-manipulating programs where updatable maps can be used to model pointers in the heap and sets to model heaplets for local reasoning [Pek et al. 2014; Reynolds 2002].

REFERENCES


A HANDLING RELATIONS IN STREAMING CONGRUENCE CLOSURE

The work in [Mathur et al. 2019a] omit relations and model them as functions. Specifically, all programs are assumed to have two fixed variables T and F (corresponding to Boolean constants true and false) that are never re-assigned. In the beginning of each program, there is an assume $\text{assume}(T \neq F)$. Further, for every $k$-ary relation $R$, there is a function $f_R$ and a variable $b_R$. Every assumption of the form “$\text{assume}(R(z))$” is translated to the sequence “$b_R := f_R(z)$”·“$\text{assume}(b_R = T)$”, and every assumption of the form “$\text{assume}(\neg R(z))$” is translated to the sequence “$b_R := f_R(z)$”·“$\text{assume}(b_R = F)$”.

This approach adds additional program variables and function symbols and further restricts the class of programs that are coherent because the memoizing restriction also applies to the newly introduced function symbols. In this paper, we show how to handle relations as first class symbols without modeling them using function symbols. For this, we will construct an automaton (similar to that in [Mathur et al. 2019a]) that accepts coherent executions (modulo the empty set of axioms $\emptyset$) iff they are feasible (modulo $\emptyset$).

Recall that executions are words over the alphabet $\Pi = \{\text{\texttt{\textasciitilde} x := y}, \text{\texttt{\textasciitilde} x := f(z)}, \text{\texttt{\textasciitilde} assume(x = y)}, \text{\texttt{\textasciitilde} assume(x \neq y)}, \text{\texttt{\textasciitilde} assume(R(z))}, \text{\texttt{\textasciitilde} assume(\neg R(z))} \mid x, y, z \text{ in } V\}.$

Let us denote by $A_{\text{SCC}}$ our automaton for streaming congruence closure. The states $Q_{\text{SCC}}$ are either the special reject state $q_{\text{reject}}$ or tuples of the form $(\equiv, d, P, \text{rel}^+, \text{rel}^-)$, where:

- $\equiv$ is an equivalence relation over $V$,
- $d$ is a symmetric and irreflexive binary relation over $V/\equiv$ (equivalence classes of $\equiv$),
- $P$ is such that for every $k$-ary function $f \in \Sigma$, $P(f)$ is a partial mapping from $(V/\equiv)^k \rightarrow V/\equiv$, and
- $\text{rel}^+$ and $\text{rel}^-$ are such that for every $k$-ary relation $R$, $\text{rel}^+(R)$ and $\text{rel}^-(R)$ are sets of $k$-tuples of $V/\equiv$ such that $\text{rel}^+(R) \cap \text{rel}^-(R) = \emptyset$.

Notice that the first three components of the state are similar to [Mathur et al. 2019a]. The later two components intuitively accumulate the relational assumes (corresponding to $y(\cdot)$ and $\delta(\cdot)$).

The transition relation $\delta_{\text{SCC}}$ of the automaton is defined as follows. Let $q = (\equiv, d, P, \text{rel}^+, \text{rel}^-)$. If $q = q_{\text{reject}}$, then $\delta_{\text{SCC}}(q, a) = q_{\text{reject}}$ for every $a \in \Pi$ (i.e., $q_{\text{reject}}$ is an absorbing state). Otherwise, we define the state $q' = \delta_{\text{SCC}}(q, a)$ as the tuple $(\equiv', d', P', \text{rel}^{'+}, \text{rel}^{''-})$ below. In each of these cases, if $d'$ becomes irreflexive or there is a relation $R$ such that $\text{rel}^{'+}(R) \cap \text{rel}^{''-}(R) \neq \emptyset$, then we set $q'$ to be $q_{\text{reject}}$.

$a = \text{\texttt{\textasciitilde} x := y}$.

Here, if $y = x$, $q' = q$. Otherwise, the variable $x$ gets updated to be in the equivalence class of $y$, and $d'$, $P'$, $\text{rel}^{'+}$ and $\text{rel}^{''-}$ are updated accordingly:

- $\equiv' = \equiv \upharpoonright_{V \setminus \{x\}} \cup \{(y', x) \mid y' \equiv y\} \cup \{(x, x)\},$
- $d' = \{(\pi[x_1], [x_2]) \mid x_1, x_2 \in V \setminus \{x\}, ([x_1]_{\equiv}, [x_2]_{\equiv}) \in d\}$
- $P'$ is such that for every $r$-ary function $h$,
  
  $$P'(h)([x_1]_{\equiv}, \ldots, [x_r]_{\equiv}) = \begin{cases} [u]_{\equiv} & x \notin \{u, x_1, \ldots, x_r\} \text{ and } \equiv' \subseteq V \setminus \{x\} \\ [u]_{\equiv} = P(h)([x_1]_{\equiv}, \ldots, [x_r]_{\equiv}) & \text{otherwise} \\ \text{undef} & \text{otherwise} \end{cases}$$

- $\text{rel}^{'+}$ is such that for every $k$-ary relation $R$,
  
  $$\text{rel}^{'+}(R) = \{(\pi[x_1], \ldots, [x_k]) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{\equiv}, \ldots, [x_k]_{\equiv} \in \text{rel}^{'+}(R))\}$$

- $\text{rel}^{''-}$ is such that for every $k$-ary relation $R$,
  
  $$\text{rel}^{''-}(R) = \{(\pi[x_1], \ldots, [x_k]) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_{\equiv}, \ldots, [x_k]_{\equiv} \in \text{rel}^{''-}(R))\}$$

\[a = "x := f(z_1, \ldots, z_k)".\]

There are two cases to consider.

1. **Case \(P(f)([z_1]_\equiv, \ldots, [z_k]_\equiv)\) is defined.**

   Let \(P(f)([z_1]_\equiv, \ldots, [z_k]_\equiv) = [v]_\equiv\). This case is similar to the case when \(a\) is "\(x := y\)". That is, when \(x \in [v]_\equiv\), then \(\equiv' = \equiv, d' = d\) and \(P' = P\). Otherwise, we have

   - \(\equiv' = \equiv \upharpoonright \{x\} \cup \{(x, v'), (v', x) \mid v' \equiv v\} \cup \{(x, x)\}\)
   - \(d' = \{([x_1]_\equiv', [x_2]_\equiv') \mid x_1, x_2 \in V \setminus \{x\}, ([x_1]_\equiv, [x_2]_\equiv) \in d\}\)
   - \(P'\) is such that for every \(r\)-ary function \(h\),
     \[
     P'(h)([x_1]_\equiv', \ldots, [x_r]_\equiv') = \begin{cases} 
     [u]_\equiv & x \notin \{u, x_1, \ldots, x_r\} \text{ and } \\
     [u]_\equiv = P(h)([x_1]_\equiv, \ldots, [x_r]_\equiv) & \text{otherwise} \\
     \text{undefined} & \text{otherwise} 
     \end{cases}
     \]

     - \(\text{rel}^{+}\) is such that for every \(k\)-ary relation \(R\),
       \[
       \text{rel}^{+}(R) = \{([x_1]_\equiv', \ldots, [x_k]_\equiv') \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_\equiv, \ldots, [x_k]_\equiv) \in \text{rel}^{+}(R)\}\]
     - \(\text{rel}^{-}\) is such that for every \(k\)-ary relation \(R\),
       \[
       \text{rel}^{-}(R) = \{([x_1]_\equiv, \ldots, [x_k]_\equiv) \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_\equiv, \ldots, [x_k]_\equiv) \in \text{rel}^{-}(R)\}\]

2. **Case \(P(f)([z_1]_\equiv, \ldots, [z_k]_\equiv)\) is undefined.**

   In this case, we remove \(x\) from its older equivalence class and make a new class that only contains the variable \(x\). We update \(P\) to \(P'\) so that the function \(f\) maps the tuple \(([z_1]_\equiv, \ldots, [z_k]_\equiv)\) (if each of them is a valid/non-empty equivalence class) to the class \([x]_\equiv\). The set \(d'\) follows easily from the new \(\equiv'\) and the older set \(d\). Thus,

   - \(\equiv' = \equiv \upharpoonright \{x\} \cup \{(x, x)\}\)
   - \(d' = \{([x_1]_\equiv, [x_2]_\equiv') \mid x_1, x_2 \in V \setminus \{x\}, ([x_1]_\equiv, [x_2]_\equiv) \in d\}\)
   - \(P'\) behaves similar to \(P\) for every function different from \(f\).
     - For every \(r\)-ary function \(h \neq f\),
       \[
       P'(h)([x_1]_\equiv', \ldots, [x_r]_\equiv') = \begin{cases} 
       [u]_\equiv & \text{if } x \notin \{u, x_1, \ldots, x_r\} \text{ and } \\
       [u]_\equiv = P(h)([x_1]_\equiv, \ldots, [x_r]_\equiv) & \text{otherwise} \\
       \text{undefined} & \text{otherwise} 
       \end{cases}
       \]
     - For the function \(f\), we have the following.
       \[
       P'(f)([x_1]_\equiv', \ldots, [x_k]_\equiv') = \begin{cases} 
       [x]_\equiv & \text{if } x_i = z_i \forall i \text{ and } x \notin \{x_1, \ldots, x_k\} \\
       [u]_\equiv & \text{if } x \notin \{u, x_1, \ldots, x_k\} \text{ and } \\
       [u]_\equiv = P(f)([x_1]_\equiv', \ldots, [x_k]_\equiv') & \text{otherwise} \\
       \text{undefined} & \text{otherwise} 
       \end{cases}
       \]
     - \(\text{rel}^{+}\) is such that for every \(k\)-ary relation \(R\),
       \[
       \text{rel}^{+}(R) = \{([x_1]_\equiv', \ldots, [x_k]_\equiv') \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_\equiv, \ldots, [x_k]_\equiv) \in \text{rel}^{+}(R)\}\]
     - \(\text{rel}^{-}\) is such that for every \(k\)-ary relation \(R\),
       \[
       \text{rel}^{-}(R) = \{([x_1]_\equiv, \ldots, [x_k]_\equiv') \mid x_1, x_2, \ldots, x_k \in V \setminus \{x\}, ([x_1]_\equiv, \ldots, [x_k]_\equiv) \in \text{rel}^{-}(R)\}\]

Here, we essentially merge the equivalence classes in which \(x\) and \(y\) belong and perform the “local congruence closure”. In addition, \(d'\) and \(P'\) are also updated as in [Mathur et al. 2019a].
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- \( \equiv' \) is the smallest equivalence relation on \( V \) such that (a) \( \equiv \cup (x, y) \subseteq \equiv' \), and (b) for every \( k \)-ary function symbol \( f \) and variables \( x_1, x'_1, x_2, x'_2, \ldots, x_k, x'_k, z, z' \in V \) such that \( [z]_\equiv = P(f)([x_1]_\equiv, \ldots, [x_k]_\equiv), [z']_\equiv = P(f)([x'_1]_\equiv, \ldots, [x'_k]_\equiv) \), and \( (x_i, x'_i) \equiv' \) for each \( i \), we have \( (z, z') \equiv' \).

- \( d' = \{(x_1)_\equiv, (x_2)_\equiv\} \setminus \{(x_1)_\equiv, (x_2)_\equiv\} \in d \)

- \( P' \) is such that for every \( r \)-ary function \( h \),

\[
P'(h)([x_1]_\equiv, \ldots, [x_r]_\equiv) = \begin{cases} [u]_\equiv' & \text{if } [u]_\equiv = P(h)([x_1]_\equiv, \ldots, [x_r]_\equiv) \\ \text{undef} & \text{otherwise} \end{cases}
\]

- \( \text{rel}'^+ \) is such that for every \( k \)-ary relation \( R \),

\[
\text{rel}'^+(R) = \{(x_1)_\equiv, \ldots, [x_k]_\equiv\} \setminus \{(x_1)_\equiv, \ldots, [x_k]_\equiv\} \in \text{rel}'^+(R))
\]

- \( \text{rel}'^- \) is such that for every \( k \)-ary relation \( R \),

\[
\text{rel}'^-(R) = \{(x_1)_\equiv, \ldots, [x_k]_\equiv\} \setminus \{(x_1)_\equiv, \ldots, [x_k]_\equiv\} \in \text{rel}'^-(R))
\]

- \( a = \text{assume}(x \neq y) \).

\[ a = \text{assume}(R(x_1, x_2, \ldots, x_k)) \]

- \( a = \text{assume}(\neg R(x_1, x_2, \ldots, x_k)) \)

We next give a proof gist for the correctness of the automaton construction. The bulk of the proof is the same as that given in [Mathur et al. 2019a]. Here, we only discuss the details necessary to prove the correctness that relates to the relational assumes.

Let us define the notion of a minimal model. Intuitively, this model has the same algebraic structure (same interpretations for constants and functions) as the initial term model as defined in [Mathur et al. 2019a]. Further, we also add relations in the minimal model on top of the initial term model. For a set of ground equalities \( A \), we will denote by \( M^\text{initial}_A \) the initial term model given by the congruence induced by \( A \).

**Definition 4.** Let \( \Gamma = \Gamma_{\text{equalities}} \cup \Gamma_{\text{relations}} \) be a set of atomic formulae of the form \( (t_1 = t_2) \in \Gamma_{\text{equalities}} \) or \( R(t_1, \ldots, t_k) \in \Gamma_{\text{relations}} \) where \( t_1, \ldots, t_k \) are ground terms over our vocabulary \( \Sigma \) and \( R \) is a \( k \)-ary relation in our vocabulary \( \Sigma \). The minimal model \( M^\text{final}_\Gamma = (U^\text{final}, \Gamma^\text{final}) \) of \( \Gamma \) is defined as follows.

- \( U^\text{min} = U^\text{initial} \)
- \( [c]^\text{min} = [c]^\text{initial} \) for \( c \in C \),
- \( [f]^\text{min} = [f]^\text{initial} \) for \( f \in F \), and
- \( [R]^\text{min} = \{(t_1)^\text{min}, \ldots, (t_k)^\text{min}\} | R(t_1, \ldots, t_k) \in \Gamma_{\text{relations}} \).
For an execution \( \rho \), the minimal model of \( \rho \) is defined by the minimal model for the set of equality and positive relational atoms in \( \kappa(\rho) \) (i.e., we do not include the dis-equality and the negative relational assumes accumulated by \( \rho \)) to define the minimal model. We will use \( M_\rho = (U_\rho, \emptyset) \) to denote this minimal model.

Notice that an execution \( \rho \) only defines a relation on the set of computed terms and thus, the minimal model never relates elements outside of the set of computed terms using relation symbols. This is formalized below.

**Lemma 17.** Let \( \rho \) be an execution and let \( M_\rho \) be the minimal model of \( \rho \). Let \( (e_1, \ldots, e_k) \in (U_\rho)^k \) be a tuple of elements in the minimal model such that one of \( e_1, \ldots, e_k \) is not computed by the execution (i.e., there is an \( 1 \leq i \leq k \) such that for every \( t \in \text{Terms}(\rho) \), \( [t]_\rho \neq e_i \)). Then, we have \( (e_1, \ldots, e_k) \notin [R]_\rho \) for every \( k \)-ary relation \( R \).

An important property about the minimal model defined above is that there is a relation preserving homomorphism from this model to any other model that satisfies the assumptions in the execution. Formally,

**Lemma 18.** Let \( M = (U_M, \emptyset) \) be a first order structure and let \( \rho \) be an execution that is feasible in \( M \). Then, there is a morphism \( h : U_\rho \to U_M \) such that

- \( h([c]_\rho) = [c]_M \) for every constant \( c \),
- \( h([f]_\rho(e_1, \ldots, e_k)) = [f]_M(h(e_1), \ldots, h(e_k)) \) for every \( k \)-ary function \( f \), and
- for every \( e_1, \ldots, e_k \in U_\rho \) and for every \( k \)-ary function, we have \( (e_1, \ldots, e_k) \in [R]_\rho \iff (h(e_1), \ldots, h(e_k)) \in [R]_M \).

Finally, we have that the minimal model is a sufficient to check for feasibility of an execution in some model (of course it is also necessary but that is evident). That is,

**Lemma 19.** Let \( \rho \) be an execution. If there is model \( M \) such that \( \rho \) is feasible in \( M \), then \( \rho \) is feasible in the minimal model \( M_\rho \).

Below, we present necessary inductive hypotheses to prove the correctness of the automaton construction. The full proof of correctness can be re-constructed using the following lemma and those used by Mathur et al in [Mathur et al. 2019a] and Lemma ??.

**Lemma 20.** Let \( \rho \) be an execution that is coherent modulo \( \emptyset \). Let \( q = (\emptyset, d, P, \text{rel}^+, \text{rel}^-) \) be the state reached after reading \( \rho \) in the automaton, i.e., \( q = \delta^*_\text{SCC}(q_0, \rho) \). If \( q \neq q_{\text{reject}} \), then we have

- for every \( x_1, x_2, \ldots, x_k \in V \) and for every \( k \)-ary relation \( R \), such that \( ([x_1]_\emptyset, [x_2]_\emptyset, \ldots, [x_k]_\emptyset) \notin \text{rel}^+(R) \), we have \( (e_1, e_2, \ldots, e_k) \notin [R]_\rho \), in the minimal model of \( \rho \), where \( e_i = [\text{TEval}(\rho, x_i)]^\text{min}_M \),
- for every \( x_1, \ldots, x_k \in V \) and for every \( k \)-ary relation \( R \), we have \( ([x_1]_\emptyset, [x_2]_\emptyset, \ldots, [x_k]_\emptyset) \in \text{rel}^-(R) \) iff for every model \( M = (U_M, \emptyset) \) for which \( \rho \) is feasible in \( M \), we have \( ([\text{TEval}(\rho, x_1)]_M, \ldots, [\text{TEval}(\rho, x_k)]_M) \notin [R]_M \).

**B PROOFS FROM SECTION 5**

**B.1 Proof of Theorem 2**

We will now prove the correctness of the reduction outlined in Fig. 3 and Fig. 4.

Let us first argue why \( P_{EPR} \) is coherent modulo the axioms \( \mathcal{A} \) in Fig. 4.

We first argue that in any execution \( \rho \) of \( P_{EPR} \), there are no equalities implied by the relational assumes. The only candidate axioms that might imply equalities are (1), (2) and (3). In any execution \( \rho \), the only relational assumes of the form \( R(t_1, t_2) \) that are implied are of the form \( R(f^n(x), f^n(y)) \)
(\(n \geq 0\)) and thus for a given \(t_1\), there is a syntactically unique \(t_2\) for which \(R(t_1, t_2)\) is implied on the computed set of terms, and thus there is no implied equality using (1) or (2). Next, the only assumptions of the form \(S(t_1, t_2)\) that are observed are of the form \(S(f^n(z), f^{n+1}(z))\) \((n \geq 0\) and \(z \in \{x, y\})\). Thus, no equality assumes are implied by (3).

Now, the only equality-assume in \(\rho\) is the one at the end of the while loop – assume\(\rho\). At the point where this assume is seen, neither \(j\) nor \(m\) have any superterms and thus there are no implied equalities due to this assume.

Let us now see why \(\rho\) is memoizing. The terms in \(j\) are always growing: \(s^n(j)\) in the \(n^{th}\) iteration. So both the assignments “\(i_{-j} := g(j)\)” and “\(j := s(j)\)” are memoizing as they never recompute terms. The same reasoning also applies to the terms in \(x\) and \(y\).

Let us now argue the correctness of the reduction.

\(\Rightarrow\). Let us assume that the given PCP instance is a YES instance. Then, there is a sequence \(i_1, i_2, \ldots, i_M\) such that \(\alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_M} = \beta_{i_1} \cdot \beta_{i_2} \cdots \beta_{i_M}\). We can now construct a model that satisfies the EPR axioms in Fig. 4 and violates the post condition. In this model, \(s\) is the successor function over \(\mathbb{N}\), \(z_0\) is the number 0 and \(g\) maps \(j\) to \(i_j\) based on the witness sequence above. Further, \(z_r\) is interpreted as the number \(r\). The variables \(x\) and \(y\) map to \(\hat{x}\) and \(\hat{y}\) respectively, which are distinct elements. The function \(f\) is such that \(f^i(\hat{x}) \neq f^j(\hat{x})\) and \(f^i(\hat{y}) \neq f^j(\hat{y})\) for every \(i \neq j \in \mathbb{N}\) and further \(f^i(\hat{x}) \neq f^j(\hat{y})\) for every \(i, j \in \mathbb{N}\). The relations \(Q_a\) are interpreted as follows: \(Q_a(f^n(\hat{x}))\) holds iff \(a\) is the \(n^{th}\) character in the sequence \(\alpha_{i_1} \cdots \alpha_{i_M}\). Similarly, \(Q_a(f^n(\hat{y}))\) holds iff \(a\) is the \(n^{th}\) character in the sequence \(\beta_{i_1} \cdots \beta_{i_M}\). Then, since \(\alpha_{i_1} \cdots \alpha_{i_M} = \beta_{i_1} \cdots \beta_{i_M}\), we must have \(R(x, y)\) at the end of the computation.

\(\Leftarrow\). In this case we have a feasible execution \(\rho\) with the statement assume \(R(x, y)\) at the end.

Consider the initial term model \(T\) for the vocabulary \(\Sigma\) (without the relations) and the starting constants \(\mathcal{V}_0 = \{x | x \in \mathcal{V}\}\). We show that it is possible to extend the term model \(T\) with interpretations of relations such that the resulting model \(T_{rels}\) is such that \(\rho\) is feasible on \(T_{rels}\). In fact, the extension is the following model: each binary and unary relation is interpreted to be the smallest relation that satisfies the assume’s in \(\rho\) as well as the EPR axioms. This is well defined because the assumes on relations in \(\rho\) are all positive assumes and all EPR axioms are monotonic, except possibly the last one, which can be handled easily: \(Q_a(t)\) holds iff \(a\) explicitly demands it. As can be seen, \(T_{rels}\) does not violate any negative assume on the relations since there are none. Further, all equality and disequality assumes are unaffected as in \(T_{rels}\), there are no terms \(t_x, t_y, t_{x_1}, t_{y_1}, t_{y_2}\) that can be instantiated for variables in the axioms (1), (2) and (3), as these relations are smallest. Thus, \(\rho\) is feasible on \(T_{rels}\).

Now from this model, we will construct the sequence \(i_1, \ldots, i_M\). The length of this sequence \(M\) will be the number of times of the while loop is executed. Clearly, the loop is executed at least once and thus \(M > 0\). Let \(t_x = f^{n_1}(\hat{x})\) and \(t_y = f^{n_1}(\hat{y})\) be the values of the variables \(x\) and \(y\) (in the term model \(T_{rels}\)). We first argue that \(n_1 = n_2\). Assume on the contrary that \(n_1 < n_2\) (w.l.o.g.). Then, one can inductively show that \(R(f^{n_1}(\hat{x}), f^{n_1}(\hat{y}))\); this is because for every \(i < n_1\), we have \(S(f^{i}(\hat{x}), f^{i+1}(\hat{x})), S(f^{i}(\hat{y}), f^{i+1}(\hat{y}))\) and also \(R(\hat{x}, \hat{y})\). But then, in the term model we have \(f^{n_1}(\hat{y}) \neq f^{n_1}(\hat{y})\) and this violates the assumption at the end of \(\rho\) (because of axiom (1)). Hence, we have \(n_1 = n_2\).

Now, the sequence \(i_1, \ldots, i_M\) can be deduced by the conditional branches in the while loop: the index \(i_j\) is the index of the branch taken in the \(j^{th}\) iteration. Let \(\alpha = \alpha_{i_1} \cdot \alpha_{i_2} \cdots \alpha_{i_M}\) and \(\beta = \beta_{i_1} \cdot \beta_{i_2} \cdots \beta_{i_M}\). First we note that \(n_1 = |\alpha|\) and \(n_2 = |\beta|\) and thus \(|\alpha| = |\beta|\). Let \(\alpha_n\) and \(\beta_n\) be
the $n^{th}$ characters of $\alpha$ and $\beta$ respectively. Then, one can see that $Q_{\alpha_n}(f^n(\hat{x}))$ and $Q_{\beta_n}(f^n(\hat{y}))$ hold in the term model. Now, axioms (5), (6) and (7) ensure that $\alpha_n = \beta_n$. Thus, $\alpha = \beta$.

### B.2 Proof of Lemma 3

We will begin by a simple technical lemma

**Lemma 21.** For $p \in \{\text{refl, irref, symm}\}$, any execution $\rho$, and any variable $x$, $\text{TEval}(\rho, x) = \text{TEval}(h^R_p(\rho), x)$.

**Proof.** The only difference between $\rho$ and $h^R_p(\rho)$ is the fact that $h^R_p(\rho)$ has additional assumes. The observation therefore, follows. \hfill $\square$

**Corollary 22.** For $p \in \{\text{refl, irref, symm}\}$ and any execution $\rho$, $\kappa(\rho) \subseteq \kappa(h^R_p(\rho))$.

**Proof.** Follows from Lemma 21. \hfill $\square$

Let $\rho$ be an arbitrary execution. We begin by first showing that $\rho$ is feasible modulo $\mathcal{A} \cup \{\phi^R_p\}$ iff $h^R_p(\rho)$ is feasible modulo $\mathcal{A}$.

Let us tackle the easy direction first. Suppose $\mathcal{M}$ is a $\mathcal{A} \cup \{\phi^R_p\}$ model such that $\rho$ is feasible in $\mathcal{M}$. Observe that since every additional assume in $h^R_p(\rho)$ holds in every $\{\phi^R_p\}$-model, we can conclude that $h^R_p(\rho)$ is feasible in $\mathcal{M}$. Also, clearly $\mathcal{M}$ is a $\mathcal{A}$-model.

Let us now prove the converse direction. Suppose $\mathcal{M} = (U_M, \llbracket \cdot \rrbracket_M)$ is a $\mathcal{A}$-model where $h^R_p(\rho)$ holds. Let $C = \{\llbracket t \rrbracket_M \mid t \in \text{Terms}(h^R_p(\rho))\}$; $C$ is the set of elements in the universe of $\mathcal{M}$ that correspond to terms that were computed in $h^R_p(\rho)$. Let $S = \{\llbracket t_1 \rrbracket_M, \llbracket t_2 \rrbracket_M \mid \rho(t_1, t_2) \in \kappa(h^R_p(\rho))\}$, which is set of pairs that correspond to terms that are assumed to be in $R$ by $h^R_p(\rho)$. Based on $\mathcal{M}$, $C$, and $S$ we will now construct a $\mathcal{A} \cup \{\phi^R_p\}$-model in which $\rho$ is feasible. This construction is case based depending on what $p$ is.

\[ p = \text{refl} \] Let $\mathcal{M}'$ be the data model that has the same universe as $\mathcal{M}$, and same interpretation for all constants, function symbols, and relation symbols other than $R$. Take $[R]_{\mathcal{M}'}$ to be $S \cup \{(e, e) \mid e \in U_M\}$. Since none of the sentences in $\mathcal{A}$ mention $R$, and $\mathcal{M}$ and $\mathcal{M}'$ have the same interpretation for everything except $R$, it immediately follows that $\mathcal{M}'$ is a $\mathcal{A}$-model. Further by definition $[R]_{\mathcal{M}'}$ is reflexive. Therefore, $\mathcal{M}'$ is a $\mathcal{A} \cup \{\phi^R_{\text{refl}}\}$-model. We need to argue that $\rho$ is feasible in $\mathcal{M}'$. First, since the interpretation of all symbols other than $R$ is the same, and $h^R_p(\rho)$ is feasible in $\mathcal{M}$, it follows that all assumes of the form $\text{assume}(c)$, where $c \not\in \{R(x, y), \neg R(x, y) \mid x, y \in V\}$, hold in $\mathcal{M}'$. Next, consider a prefix of $\rho$ the form $\pi \cdot \text{assume}(R(x, y))$. Since $\kappa(\rho) \subseteq \kappa(h^R_{\text{refl}}(\rho))$, $[t]_{\mathcal{M}'} = [t]_{\mathcal{M}}$ for any term $t$, and $S \subseteq [R]_{\mathcal{M}'}$, we have $(\text{eval}_{\mathcal{M}'}(\pi, x), \text{eval}_{\mathcal{M}'}(\pi, y)) \in [R]_{\mathcal{M}'}$. Let us now consider a prefix of $\rho$ of the form $\pi \cdot \text{assume}(\neg R(x, y))$. Observe since $h^R_{\text{refl}}(\rho)$ is feasible in $\mathcal{M}$, $\rho(t, t) \in \kappa(h^R_{\text{refl}}(\rho))$ for every $t \in \text{Terms}(h^R_{\text{refl}}(\rho))$, and $h^R_{\text{refl}}(\pi) \cdot \text{assume}(\neg R(x, y))$ is a prefix of $h^R_{\text{refl}}(\rho)$, it must be the case that $\text{eval}_{\mathcal{M}'}(\pi, x) = \text{eval}_{\mathcal{M}'}(h^R_{\text{refl}}(\rho), x) \neq \text{eval}_{\mathcal{M}'}(h^R_{\text{refl}}(\rho), y) = \text{eval}_{\mathcal{M}'}(\pi, y)$. Further, we have $S \subseteq [R]_{\mathcal{M}'} \cap (C \times C)$ and therefore, if $\text{eval}_{\mathcal{M}'}(\pi, x) \neq \text{eval}_{\mathcal{M}'}(\pi, x)$ then this assume also holds in $\mathcal{M}'$.

\[ p = \text{irref} \] Let $\mathcal{M}'$ to be the data model that has the same universe as $\mathcal{M}$ and the same interpretation for constants, functions, and relations other than $R$. In $\mathcal{M}'$, we take $S$ to be the interpretation for $R$. By reasoning similar to the case for reflexivity, $\mathcal{M}'$ can be seen to be a $\mathcal{A}$-model. We will now argue that $\mathcal{M}'$ is $\{\phi^R_{\text{irref}}\}$-model, i.e., $S \cap \{(e, e) \mid e \in U_M\} = \emptyset$. Suppose for some $e, (e, e) \in S$. Note that $S \subseteq [R]_{\mathcal{M}}$, and so $(e, e) \in [R]_{\mathcal{M}}$. If $(e, e) \in S$ then there are terms $t_1, t_2 \in \text{Terms}(h^R_{\text{irref}}(\rho))$ such that $\llbracket t_1 \rrbracket_M = e = \llbracket t_2 \rrbracket_M$. From the definition of $h^R_{\text{irref}}$, we know
that for every term \( t \in \text{Terms}(h^R_{\text{irref}}(\rho)) \), \( \neg R(t, t) \in \kappa(h^R_{\text{irref}}(\rho)) \). But then \( h^R_{\text{irref}}(\rho) \) cannot be feasible in \( M \) as \((e, e) = ([t_1]_M, [t_1]_M) \in [R]_M \) and \( \neg R(t_1, t_1) \in \kappa(h^R_{\text{irref}}(\rho)) \). We get the desired contradiction to establish that \( M' \) is a \( \{\varphi_{\text{irref}}\} \)-model. To prove that \( \rho \) is feasible in \( M' \) we need to show that all \textit{assumes} in \( \rho \) hold in \( M' \). Since \( M \) and \( M' \) agree on interpretations of everything except \( R \), and Corollary 22, all assumes not involving \( R \) hold in \( M' \). Next, as \( S \subseteq [R]_M \), all \textit{assume} \((\neg R(x, y))\) also hold in \( M' \). Finally, a \textit{assume} \((R(x, y))\) holds by definition of \( S \). Thus, \( \rho \) is feasible in \( M' \).

\( p = \text{symm} \) Again we take \( M' \) to be the model identical to \( M \) except the interpretation of \( R \) is taken to be \( S \). Clearly, \( M' \) is a \( \mathcal{A} \)-model since sentences in \( \mathcal{A} \) do not mention \( R \). Further, by definition of \( h^R_{\text{symm}} \), for every \( t_1, t_2 \in \text{Terms}(h^R_{\text{symm}}(\rho)) \), if \( R(t_1, t_2) \in \kappa(h^R_{\text{symm}}(\rho)) \) then \( R(t_2, t_1) \in \kappa(h^R_{\text{symm}}(\rho)) \). Therefore \( S \) is symmetric, and so \( M' \) is a \( \{\varphi_{\text{symm}}\} \)-model. Finally, feasibility of \( \rho \) in \( M' \) holds for the following reasons: (a) Assumes not involving \( R \) hold because \( M \) and \( M' \) only disagree on \( R \), \( h^R_{\text{symm}}(\rho) \) is feasible in \( M \), and \( \kappa(\rho) \subseteq \kappa(h^R_{\text{symm}}(\rho)) \); (b) Assumes of \( \neg R \) hold because \( S \subseteq [R]_M \); and (c) Assumes of \( R \) hold because of the definition of \( S \).

To prove coherence preservation, we will find the following lemma useful to establish.

Lemma 23. Let \( \mathcal{A} \) be a set of first order sentences such that the relation symbol \( R \) does not syntactically appear in any sentence in \( \mathcal{A} \). Let \( \rho \in \{\text{refl}, \text{irref}, \text{symm}\} \). For any execution \( \rho \), and any two computed terms \( t_1, t_2 \in \text{Terms}(h^R_{\rho}(\rho)) \),

\[
 t_1 \equiv_{\mathcal{A} \cup \{\varphi^R_{\rho}\} \cup \kappa(\rho)} t_2 \quad \text{iff} \quad t_1 \equiv_{\mathcal{A} \cup \kappa(h^R_{\rho}(\rho))} t_2.
\]

Proof. First observe that for every \( \psi \in \kappa(h^R_{\rho}(\rho)) \setminus \kappa(\rho) \), we have \( \mathcal{A} \cup \{\varphi^R_{\rho}\} \models \psi \). Therefore, every \( \mathcal{A} \cup \{\varphi^R_{\rho}\} \kappa(\rho) \)-model is also a \( \mathcal{A} \cup \kappa(h^R_{\rho}(\rho)) \)-model. Hence, if \( t_1 \equiv_{\mathcal{A} \cup \kappa(h^R_{\rho}(\rho))} t_2 \) then \( t_1 \equiv_{\mathcal{A} \cup \{\varphi^R_{\rho}\} \cup \kappa(\rho)} t_2 \).

For the other direction, suppose \( t_1 \not\equiv_{\mathcal{A} \cup \{\varphi^R_{\rho}\} \cup \kappa(\rho)} t_2 \). Then by definition, there is a \( \mathcal{A} \cup \kappa(h^R_{\rho}(\rho)) \)-model \( M \) such that \( [t_1]_M \neq [t_2]_M \). Consider the execution \( \rho_1 = \rho \cdot \text{"assume"}(t_1 \neq t_2) \). Technically \( \rho_1 \) is not an execution by our definition. What we mean is to copy the terms \( t_1 \) and \( t_2 \) in fresh variables when they are computed, and assume that those variables are not equal; we skip doing this precisely. Observe that \( h^R_{\rho}(\rho_1) = h^R_{\rho}(\rho) \cdot \text{"assume"}(t_1 \neq t_2) \). Based on our assumptions, \( h^R_{\rho}(\rho_1) \) is feasible in \( M \). By our observation that feasibility is preserved, we have \( \rho_1 \) is feasible in some \( \mathcal{A} \cup \{\varphi^R_{\rho}\} \)-model \( M' \). Thus, \( \{[t_1]_M \neq [t_2]_M \} \cdot \text{assert}(t_1 \neq t_2) \), and so \( t_1 \not\equiv_{\mathcal{A} \cup \{\varphi^R_{\rho}\} \cup \kappa(\rho)} t_2 \). \( \square \)

We are now ready to prove that coherence is preserved. Suppose \( \rho \) is coherent modulo \( \mathcal{A} \cup \{\varphi^R_{\rho}\} \). Consider any \( \pi \cdot \text{"x:=f(x)"} \) prefix of \( h^R_{\rho}(\rho) \). Notice that based on the definition of \( h^R_{\rho} \) (no matter what \( \rho \) is), there is a prefix \( \pi' \) of \( \rho \) such that \( h^R_{\rho}(\pi') = \pi \) and \( \pi' \cdot \text{"x:=f(x)"} \) is a prefix of \( \rho \). Using Lemma 23 on \( \pi' \) and \( \pi \), and using the fact \( \rho \) is memoizing, we can argue that the prefix \( \pi \) is also memoizing. Consider a prefix \( \pi \cdot \text{"assume"}(\cdot) \) of \( h(\rho) \). If there is a prefix \( \pi' \) of \( \rho \) such that \( h^R_{\rho}(\pi') = \pi \) and \( \pi' \cdot \text{"assume"}(\cdot) \) is a prefix of \( \rho \) (i.e., this is not a new assume added by \( h^R_{\rho} \), then the fact that \textit{assume}(\cdot) \) is early follows from Lemma 23 and the fact that this is an early assume in \( \rho \). On the other hand, if this is a new assume then there is a prefix \( \pi' \) of \( \rho \) and \( a \) such that \( \pi' \cdot a \) is a prefix of \( \rho \), and \( h^R_{\rho}(\pi' \cdot a) = \pi_1 \). We now consider different cases based on \( \rho \).

\( \rho \in \{\text{refl}, \text{irref}\} \) In this case \( a \) is of the form \( x := f(x) \). Let \( \pi_1 = \pi \cdot \text{"assume"}(\cdot) \); we have \( h^R_{\rho}(\pi' \cdot a) = \pi_1 \). Suppose \( t_1 \equiv_{\mathcal{A} \cup \kappa(\pi_1)} t_2 \). Then by Lemma 23, \( t_1 \equiv_{\mathcal{A} \cup \{\varphi^R_{\rho}\} \cup \kappa(\pi' \cdot a)} t_2 \). Because \( a \) is an assignment step, \( \kappa(\pi' \cdot a) = \kappa(\pi') \). Therefore, \( t_1 \equiv_{\mathcal{A} \cup \{\varphi^R_{\rho}\} \cup \kappa(\pi')} t_2 \). Again using Lemma 23, we have \( t_1 \equiv_{\mathcal{A} \cup \kappa(h^R_{\rho}(\pi'))} t_2 \), which proves that this is an early assume.

\( p = \text{symm} \) In this case \( a \) is of the form \( \text{assume}(R(x, y)) \). Again take \( \pi_1 = \pi \cdot "\text{assume}(c)" \), and we have \( h^R_p(\pi \cdot a) = \pi_1 \). Suppose \( t_1 \equiv_{\mathcal{A} \cup \{ h^R_p(\pi \cdot a) \}} t_2 \). Then by Lemma 23, \( t_1 \equiv_{\mathcal{A} \cup \{ h^R_p(\pi \cdot a) \}} t_2 \).

Suppose one of \( t_1 \) or \( t_2 \) is dropped in \( \pi_1 \), then by Lemma 23, we know that it is also dropped in \( \pi \cdot a \). Since \( a \) is an early assume, in that case we have \( t_1 \equiv_{\mathcal{A} \cup \{ h^R_p(\pi \cdot a) \}} t_2 \). Using Lemma 23 again, we have \( t_1 \equiv_{\mathcal{A} \cup \{ h^R_p(\pi \cdot a) \}} t_2 \), which proves that the assume is early.

We now establish the other direction of coherence preservation. Suppose \( h^R_p(\rho) \) is coherent modulo \( \mathcal{A} \). Consider a prefix of the form \( \pi \cdot "x := f(z)" \) of \( \rho \). By definition of \( h^R_p \), \( h^R_p(\pi \cdot "x := f(z)") \) is also memoizing in \( h^R_p(\rho) \), by Lemma 23, it is also memoizing in \( \rho \). Consider prefix \( \pi \cdot "\text{assume}(c)" \) of \( \rho \). Now no matter what \( \rho \) is, \( h^R_p(\pi) \cdot "\text{assume}(c)" \) is a prefix of \( h^R_p(\pi \cdot "\text{assume}(c)" ) \) (and hence also of \( h^R_p(\rho) \)). Since \( \text{assume}(c) \) is early in \( h^R_p(\rho) \), using Lemma 23, we can conclude that it is also early in \( \rho \). This \( \rho \) is coherent modulo \( \mathcal{A} \cup \{ h^R_p \} \).

### B.3 Proof of Theorem 5

In this section, we prove coherence modulo transitivity is decidable. More precisely, let \( \mathcal{R}_{\text{trans}} \) be the set of binary relations that are transitive and let \( \mathcal{A}_{\text{trans}} = \{ \phi^R_{\text{trans}} \mid R \in \mathcal{R}_{\text{trans}} \} \). We will show that the set CoherentExecs(\( \Sigma, V, \mathcal{A}_{\text{trans}} \)) is a regular language. For this, we modify the automaton construction in Appendix A to accommodate transitive relations.

The states of the automaton are still the same as that described in Appendix A. Further, the transition function \( \delta_{\text{SCC}} \) is such that for a state \( q \neq q_{\text{reject}} \), \( \delta_{\text{SCC}}(q, a) \) is the same as before when \( a \notin \{ "\text{assume}(R(x, y))", "\text{assume}(\neg R(x, y))" \mid R \in \mathcal{R}_{\text{trans}} \} \). Below we give the modified transitions for these cases.

The intuitive idea behind the modification is as follows. For \( R \in \mathcal{R}_{\text{trans}} \) component \( \text{rel}^+(R) \) stores the pairs of equivalence classes which are implied by the transitive closure of the observed assume statements "\( \text{assume}(R(x, y)) \)". For example, if the execution observes "\( \text{assume}(R(x, y)) \)" and "\( \text{assume}(R(y, z)) \)" then the component \( \text{rel}^+(R) \) stores the pair \( ([x]_=, [y]_=) \) in addition to the pairs \( ([x]_=, [z]_=) \) and \( ([y]_=, [z]_=) \). Next, for every \( R \in \mathcal{R}_{\text{trans}} \), the component \( \text{rel}^{-}(R) \) also adds additional pair \( (c_1, c_2) \) of equivalence classes for which \( \neg R(c_1, c_2) \) is implied by the positive and negative assumes in the execution. More precisely, if the execution observes \( \text{assume}(R(x, y)) \) and \( \text{assume}(\neg R(x, y)) \) then one can infer the constraint \( \neg R(y, z) \), and in this case, we also add \( ([y]_=, [z]_=) \) in \( \text{rel}^-(R) \) in addition to \( ([x]_=, [z]_=) \). Similarly, if the execution observes \( \text{assume}(R(y, z)) \) and \( \text{assume}(\neg R(y, z)) \) then one can infer the constraint \( \neg R(x, y) \), and in this case, we also add \( ([y]_=, [z]_=) \) in \( \text{rel}^-(R) \) in addition to \( ([x]_=, [z]_=) \).

Let us now give the formal definition of \( \delta_{\text{SCC}}(q, a) \) when \( q \neq q_{\text{reject}} \) and when \( a \in \{ "\text{assume}(R(x, y))", "\text{assume}(\neg R(x, y))" \mid R \in \mathcal{R}_{\text{trans}} \} \). As before, if \( \text{rel}^+(R) \cap \text{rel}^-(R) \neq \emptyset \), we go to the state \( q_{\text{reject}} \).

\[
\begin{align*}
a &= "\text{assume}(R(x, y))".
\end{align*}
\]

In this case, \( \equiv = \equiv \), \( P' = P \), \( d' = d \). Further, \( \text{rel}^+(R') = \text{rel}^+(R') \) and \( \text{rel}^-(R') = \text{rel}^-(R') \) for every \( R' \neq R \). Further,

- \( \text{rel}^+(R) \) is the smallest set such that
  - (a) \( \text{rel}^+(R) \subseteq \text{rel}^+(R) \), and
  - (b) is transitively closed, i.e., for all \( x, y, z \in V \) if \( ([x]_=, [y]_=) \in \text{rel}^+(R) \) and \( ([y]_=, [z]_=) \in \text{rel}^+(R) \) then \( ([x]_=, [z]_=) \in \text{rel}^+(R) \).
- \( \text{rel}^+(R) \) is the smallest set such that
  - (a) \( \text{rel}^+(R) \subseteq \text{rel}^+(R) \), and
  - (b) for all \( x, y, z \in V \) if \( ([x]_=, [y]_=) \in \text{rel}^+(R) \) and \( ([y]_=, [z]_=) \in \text{rel}^+(R) \) then \( ([y]_=, [z]_=) \in \text{rel}^+(R) \), and
(c) for all \(x, y, z \in V\) if \(([x], [y]) \in \text{rel}^+ (R)\) and \(([x], [z]) \in \text{rel}^- (R)\) then \(([y], [z]) \in \text{rel}^+ (R)\).

\[ a = \text{assume}(\neg R(x, y))\].

In this case, \(R = \equiv, P' = P, d' = d\) and \(\text{rel}^+ = \text{rel}^+\) and \(\text{rel}^- (R') = \text{rel}^- (R')\) for every \(R' \neq R\).

Further, \(\text{rel}^+ (R)\) is the smallest set such that

(a) \(\text{rel}^- (R) \subseteq \text{rel}^- (R')\), and

(b) for all \(x, y, z \in V\) if \(([x], [y]) \in \text{rel}^+ (R)\) and \(([x], [z]) \in \text{rel}^- (R)\) then \(([y], [z]) \in \text{rel}^+ (R)\), and

(c) for all \(x, y, z \in V\) if \(([x], [y]) \in \text{rel}^+ (R)\) and \(([x], [z]) \in \text{rel}^- (R)\) then \(([y], [z]) \in \text{rel}^+ (R)\).

In order to argue correctness, we extend the notion of minimal model to transitivity.

**Definition 5.** Let \(\mathcal{A}_{\text{trans}}\) be the set of transitivity axioms on some finite set of binary relations \(\mathcal{R}_{\text{trans}}\). Let \(\rho\) be an execution and let \(A\) be the set of equalities in \(\kappa(\rho)\). Let \(M_{\rho}\) be the minimal model for \(\rho\). Define the minimal transitive model (with respect to \(\mathcal{R}_{\text{trans}}\)) of \(\rho\) to be the model

\[
M^{\text{trans}}_{\rho} = (U^{\text{trans}}_{\rho}, \llbracket\mathcal{A}_{\text{trans}}\rrbracket_{\rho})\]

such that \(U^{\text{trans}}_{\rho} = U_{\rho}, \llbracket c \rrbracket_{\rho}^{\text{trans}} = \llbracket c \rrbracket_{\rho}\) for every \(c \in C, \llbracket f \rrbracket_{\rho}^{\text{trans}} = \llbracket f \rrbracket_{\rho}\) for every \(f \in \mathcal{T}\) and \(\llbracket R \rrbracket_{\rho}^{\text{trans}} = [R]_{\rho}\) for every \(R \in \mathcal{R} \setminus \mathcal{R}_{\text{trans}}\). Further, for every \(R \in \mathcal{R}_{\text{trans}}\), define \(\llbracket R \rrbracket_{\rho}^{\text{trans}}\) to be the smallest transitive set containing \(\llbracket R \rrbracket_{\rho}\).

Notice that the execution \(\rho\) only defines a relation on the set of computed terms, and thus the transitive closure of the observed assumes also stays with the set of computed terms. This is formalized below.

**Lemma 24.** Let \(\rho\) be an execution and let \(M^{\text{trans}}\) be the minimal transitive model as defined above. Let \(e_1, e_2 \in U^{\text{trans}}_{\rho}\) be elements in the minimal model such that either \(e_1\) or \(e_2\) is not computed by the execution (i.e., there is an \(i \in \{1, 2\}\) such that for every \(t \in \text{Terms}(\rho), \llbracket t \rrbracket_{\rho}^{\text{trans}} \neq e_i\)). Then, we have \((e_1, e_2) \notin \llbracket R \rrbracket_{\rho}^{\text{trans}}\).

An important property about the minimal transitive model defined above is that there is a relation preserving homomorphism from this model to any other model that satisfies the assumptions in the execution and the transitivity axioms. Formally,

**Lemma 25.** Let \(M = (U_M, \llbracket\mathcal{A}_{\text{trans}}\rrbracket)\) be a first order model and let \(\rho\) be an execution that is feasible in \(M\), modulo \(\mathcal{A}_{\text{trans}}\). Then, there is a morphism \(h : U^{\text{trans}}_{\rho} \to U_M\) such that

- \(h(\llbracket f \rrbracket_{\rho}^{\text{trans}}(e_1, \ldots, e_k)) = \llbracket f \rrbracket_M(h(e_1), \ldots, h(e_k))\) for every \(k\)-ary function \(f\), and
- for every \(e_1, \ldots, e_k \in U^{\text{trans}}_{\rho}\) and for every \(k\)-ary function, we have

\[(e_1, \ldots, e_k) \in \llbracket R \rrbracket_{\rho}^{\text{trans}} \implies (h(e_1), \ldots, h(e_k)) \in \llbracket R \rrbracket_M\]

Finally, we have that the minimal model is enough to check for feasibility of an execution in some model. That is,

**Lemma 26.** Let \(\rho\) be an execution that is feasible in \(M\). Let \(\mathcal{A}_{\text{trans}}\) be the set of transitivity axioms for relations in \(\mathcal{R}_{\text{trans}}\). If there is model \(\check{M}\) such that \(\rho\) is feasible in \(\check{M}\), then \(\rho\) is feasible in the minimal model \(M^{\text{trans}}_{\rho}\).

We prove the correctness of the automaton construction by inducting on the length of the word. For this, we will be using the following inductive invariants.

**Lemma 27.** Let \(\mathcal{A}_{\text{trans}}\) be the set of transitivity axioms on some finite set of binary relations \(\mathcal{R}_{\text{trans}}\). Let \(\rho\) be an execution that is coherent modulo \(\mathcal{A}_{\text{trans}}\). Let \(q = (\equiv, d, P, \text{rel}^+, \text{rel}^-)\) be the state reached after reading \(\rho\) in the automaton, i.e., \(q = \delta^*_{\text{SCC}}(q_0, \rho)\). If \(q \neq q_{\text{reject}}\), then we have \((\text{here } R \in \mathcal{R}_{\text{trans}})\).
for every $x, y \in V$ such that $([x]_\rho, [y]_\rho) \notin \text{rel}^\tau(R)$, we have $(e_x, e_y) \notin \text{Eval}^\rho(R)$, in the minimal model of $\rho$, where $e_x = \text{Eval}(\rho, x)$ and $e_y = \text{Eval}(\rho, y)$.

• for every $x, y \in V$, $([x]_\rho, [y]_\rho) \in \text{rel}^\tau(R)$ iff for every model $M = (U_M, \llbracket M \rrbracket)$ for which $\rho$ is feasible in $M$, we have $(\text{Eval}(\rho, x), \text{Eval}(\rho, y)) \notin \text{Eval}(\rho, M)$.

C PROOFS FROM SECTION 6

C.1 Proof of Theorem 11

We prove that the execution $\rho$ shown in Fig. 6 is coherent and the reduction is correct.

Let us first argue why $\rho_{assoc}$ is coherent modulo associativity of $f$. This follows because all created terms are being retained in some program variables.

Now, we argue the correctness of the reduction.

($\Leftarrow$). Assume that the given instance of the word problem is a NO instance. Then, there is a semi group $(A, \circ)$ and a homomorphism $h : S \rightarrow A$ such that for each $1 \leq i \leq N$, $h(u_i) = h(v_i)$ and $h(u_0) \neq h(v_0)$. Then, the model $M = (U_M, \llbracket M \rrbracket)$ with $U_M = A$ and $\llbracket f \rrbracket_M = \circ$ is the model on which $\rho_{assoc}$ is feasible. Further, $\llbracket f \rrbracket_M$ is associative and all the assumptions in $\rho_{assoc}$ hold in the model.

($\Rightarrow$). Assume the execution $\rho_{assoc}$ is feasible modulo associativity. That is, there is a model $M = (U_M, \llbracket M \rrbracket)$ such that $\llbracket f \rrbracket_M$ is associative and all the assumptions in the execution are true in the model. Then, clearly $U_M$ with $\llbracket f \rrbracket_M$ as concatenation is a semigroup. Further, there is a homomorphism $h$ from $S$ to $A = (U_M, \llbracket M \rrbracket)$ given by $h(a_i) = \llbracket a_i \rrbracket_M$ for every $a_i \in \Delta$. Since the string $u_0$ and $v_0$ are not equal in $A$, the equality $u_0 = v_0$, the given instance is a NO instance of the word problem.
on which \( \sigma \) is feasible, restricted to the set of computed terms in \( \sigma \). Now, once we have that \( \sigma \) is feasible on \( M_{\mathcal{A}} \), we will argue that in fact \( \rho \) itself is also feasible on \( M_{\mathcal{A}} \).

We construct a minimal model \( M_{\mathcal{A}} = (U_{\mathcal{A}}, [\cdot]_{\mathcal{A}}) \) for \( \mathcal{A} \) as follows. The algebraic structure of \( M_{\mathcal{A}} \) is the same as the canonical term algebra \( M_{\text{init}} = (U_{\text{init}}, [\cdot]_{\text{init}}) \) given by (a) the commutativity axioms, (b) idempotence axioms and (c) equality assumes in \( \rho \). We set, \( U_{\mathcal{A}} = U_{\text{init}}, [c]_{\mathcal{A}} = [c]_{\text{init}} \) for every \( c \in C \) and \( [f]_{\mathcal{A}} = [f]_{\text{init}} \) for every \( f \in \mathcal{F} \). For a relation \( R \), let us denote by \( S_R \) the set \( \{ ([t_1]_{\mathcal{A}}, \ldots, [t_k]_{\mathcal{A}}) \mid R(t_1, \ldots, t_k) \in \kappa(\rho) \} \). For an uninterpreted relation \( R \), define \( [R]_{\mathcal{A}} = S_R \). For a reflexive relation \( R \), define \( [R]_{\mathcal{A}} \) to be the reflexive closure of \( S_R \). For a symmetric relation \( R \), define \( [R]_{\mathcal{A}} \) to be the symmetric closure of \( S_R \). For a relation that is both reflexive and symmetric, define \( [R]_{\mathcal{A}} \) to be the reflexive, symmetric closure of \( S_R \). We additionally close \( [R]_{\mathcal{A}} \) transitively if \( R \) is transitive.

Now let \( M = (U_M, [\cdot]_M) \) be some model on which \( \sigma = h(\rho) \) is feasible. Let \( T_\sigma = \text{Terms}(\sigma) \) be the set of terms computed in \( \sigma \). We show that there is a morphism \( m : M_{\mathcal{A}} \rightarrow M \) such that

1. for every term \( t \in T_\sigma \), we have \( m([t]_{\mathcal{A}}) = [t]_M \), and
2. for every relation \( R \in \mathcal{R} \) of arity \( r \) and every \( r \)-tuple \( (t_1, \ldots, t_r) \in T_\sigma^r \), we have \( ([t_1]_{\mathcal{A}}, \ldots, [t_r]_{\mathcal{A}}) \in [R]_{\mathcal{A}} \iff ([t_1]_M, \ldots, [t_r]_M) \in [R]_M \)

Condition (1) follows from the fact that, on the set of terms \( T_\sigma \), \( M \) at least satisfies all the commutativity and idempotence axioms, and also all the equalities in \( \rho \). This is ensured using our instrumentation \( h \). Condition (2) follows from the fact that, for the set of computed set of terms, \( \sigma \) ensures reflexivity, irreflexivity and symmetry of different axioms.

Now because of the existence of the above morphism and the fact that all assumes in \( \sigma \) are over the terms computed, we have that \( \sigma \) is feasible iff it is feasible on \( M_{\mathcal{A}} \).

Now let us show that \( M_{\mathcal{A}} \) is a model of \( \rho \). This is because all the assumptions in \( \rho \) were also present in \( \sigma \) and \( \sigma \) is feasible on \( M_{\mathcal{A}} \). Additionally, observe that \( M_{\mathcal{A}} \) satisfies all the axioms in \( \mathcal{A} \). Thus, \( \rho \) is feasible modulo \( \mathcal{A} \).