Exact Quantitative Probabilistic Model Checking Through Rational Search

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Probabilistic Systems Everywhere

Systems exhibiting stochastic behavior :

- Modeling unreliable/unpredictable behavior - processor failure, message loss, etc.,
- Model-based performance evaluation - analyze average waiting time, delay queue length, etc.,
- Cryptographic protocols and encryption schemes privacy and security
- Distributed and network protocols - break symmetry

Modeling Probabilistic Systems

State-transition systems with probabilistic transitions, remain the popular choice for modeling probabilistic systems.

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Discrete Time Markov Chains

A DTMC is a tuple $\mathcal{M} = (Z, \Delta, L)$, where

- Z is a finite set of states
- $\Delta : Z \to \text{Dist}(Z)$ is the probabilistic transition function of $\mathcal{M}$
- $L : Z \to 2^{\text{AP}}$ is a labeling function

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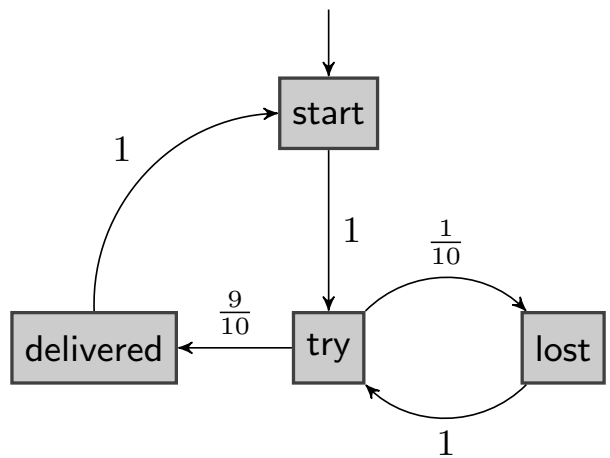

Figure: DTMC modeling a simple communication protocol

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Verification of probabilistic systems often involves checking the probability of reaching some set of states.

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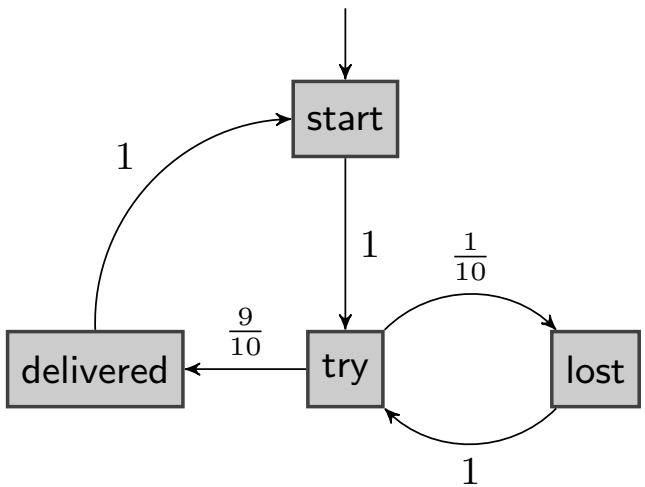

Probability of reaching the state 'delivered' starting from each state ?

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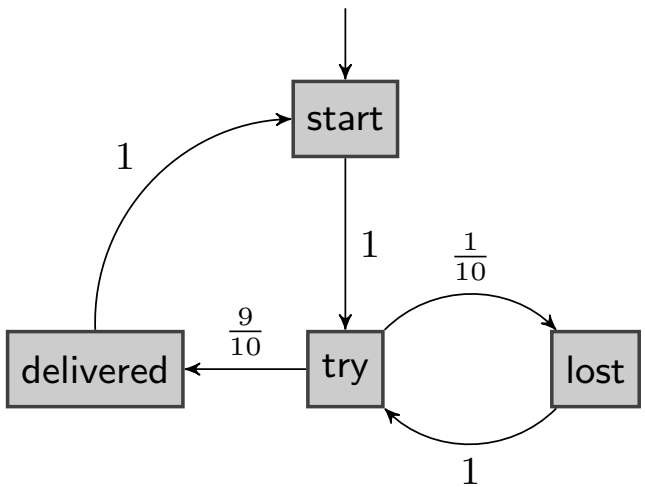

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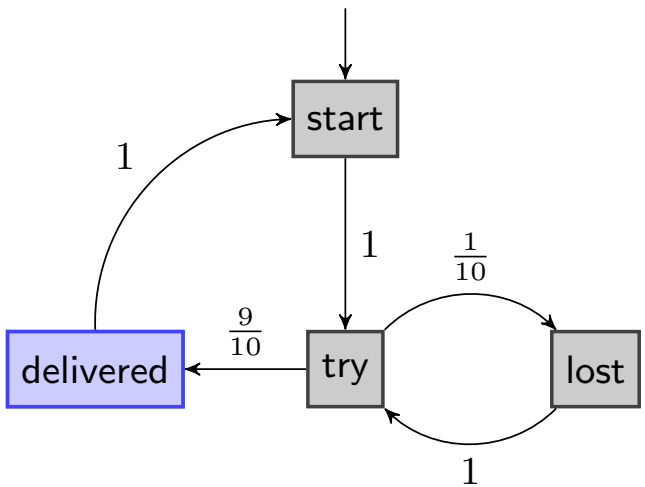

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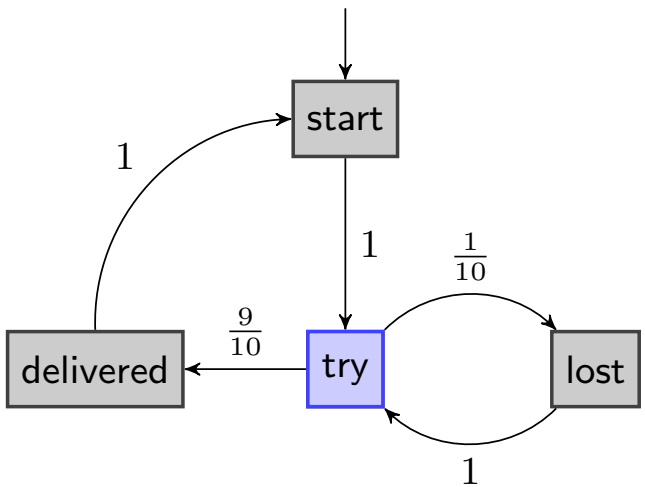

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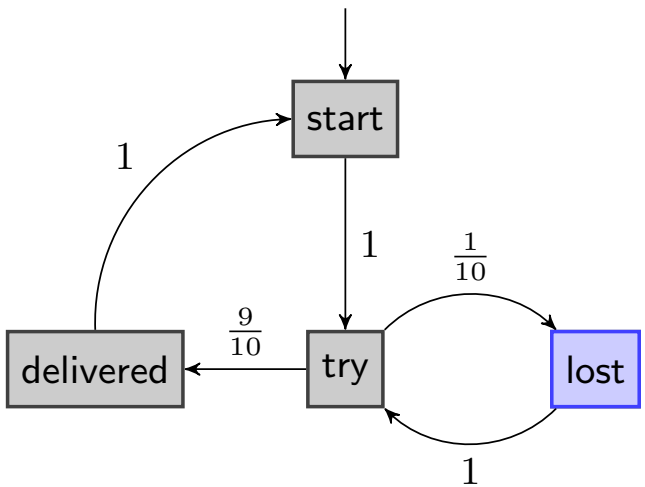

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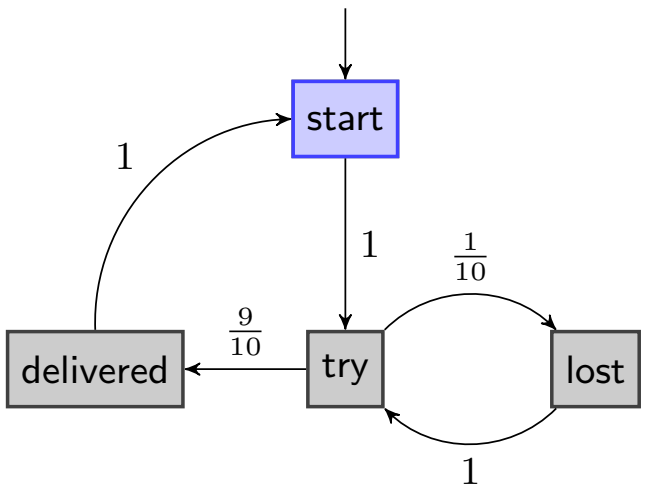

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  - rarely used in practice for quantitative model checking
- State-of-the-art model checkers, such as PRISM, resort to techniques that compute approximations

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- This technique is called value iteration

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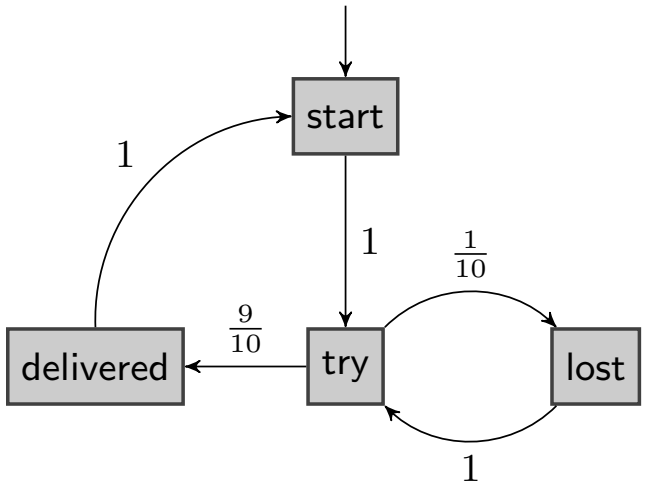

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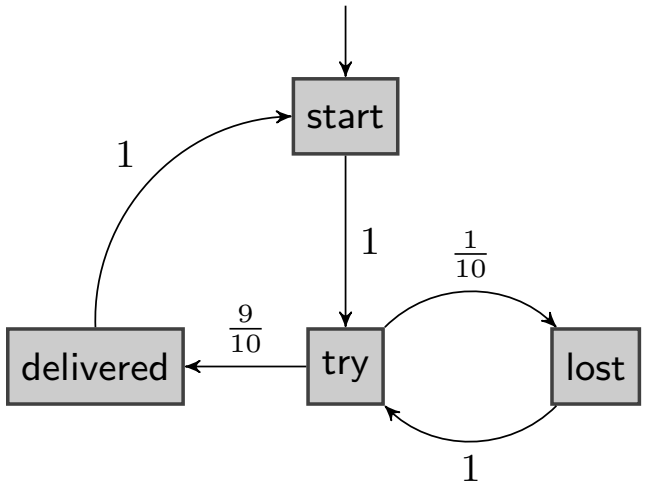

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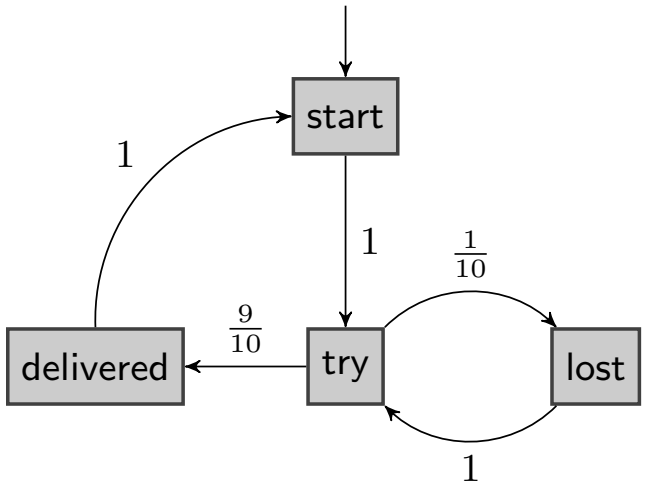

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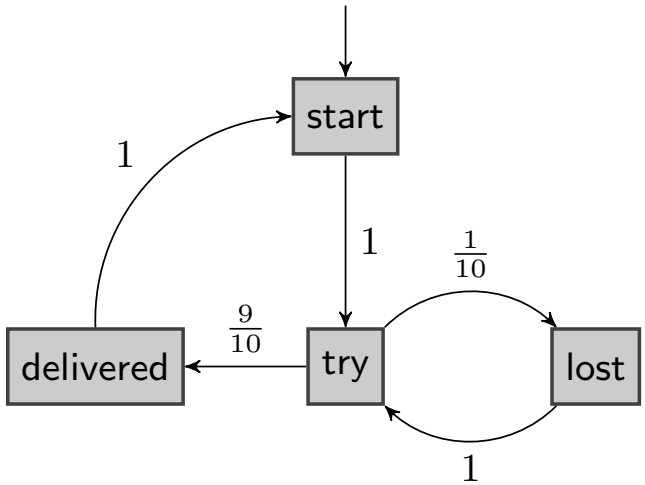

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x_{start}	0
x_{try}	0
x_{lost}	0
$x_{\text{delivered}}$	0

Init

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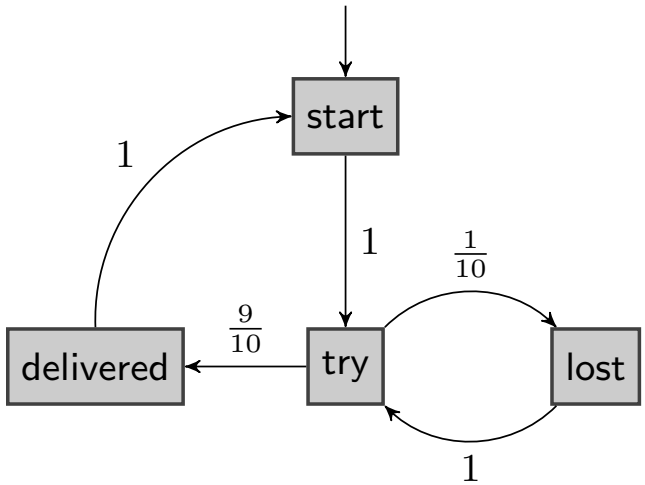

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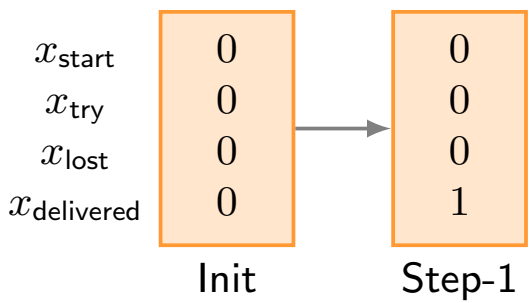

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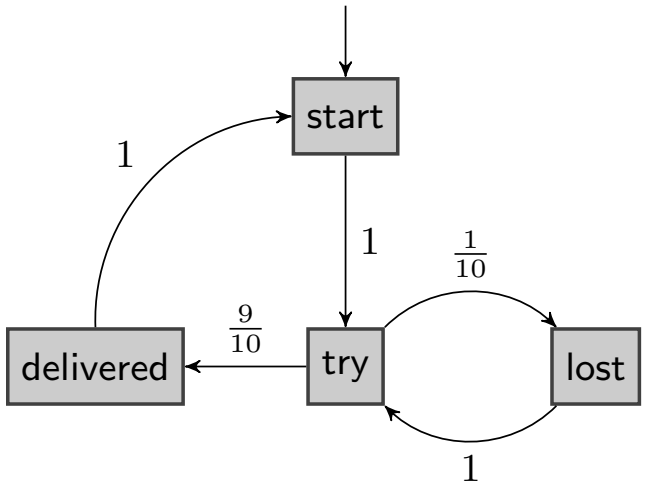

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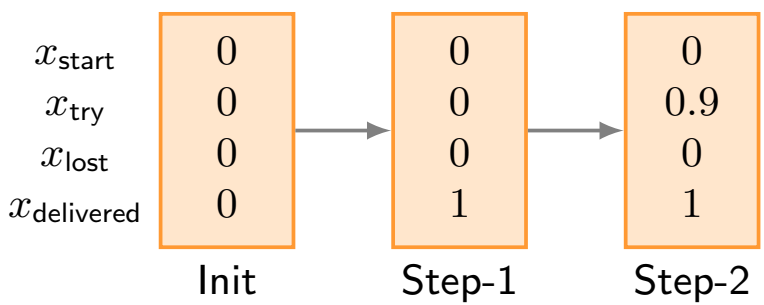

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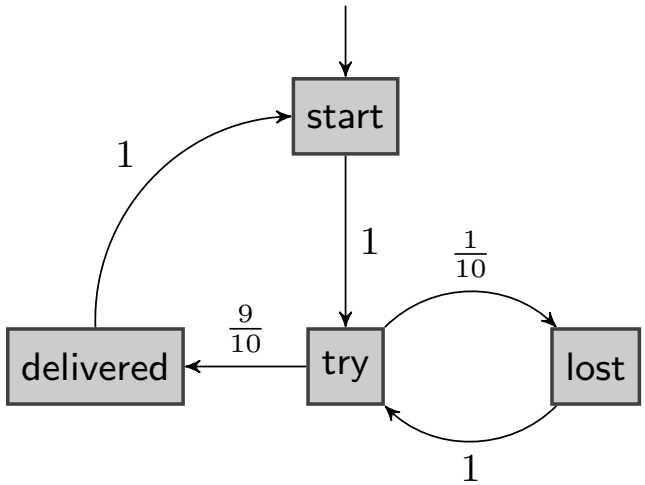

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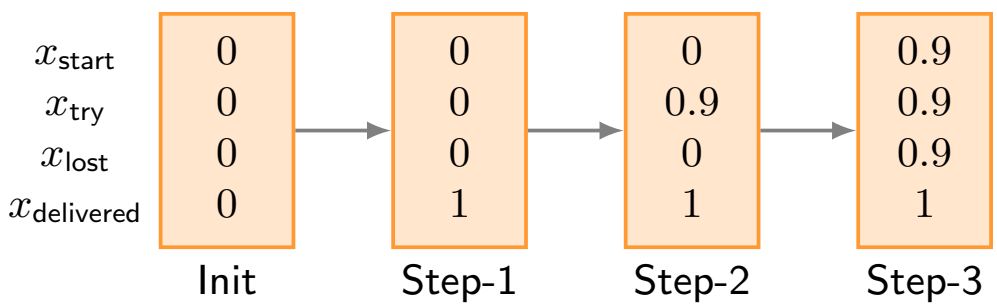

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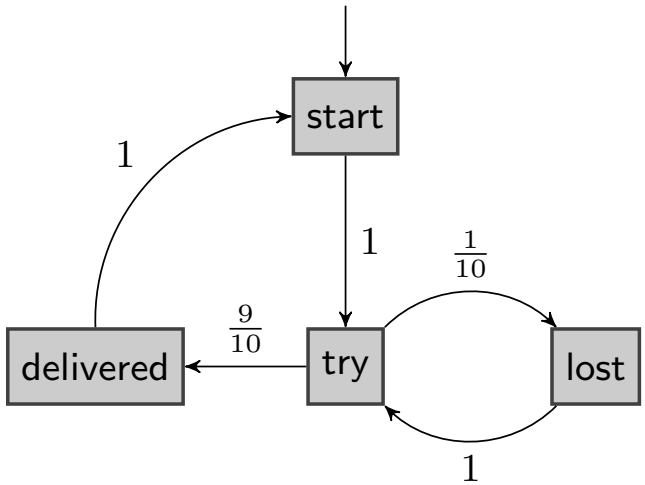

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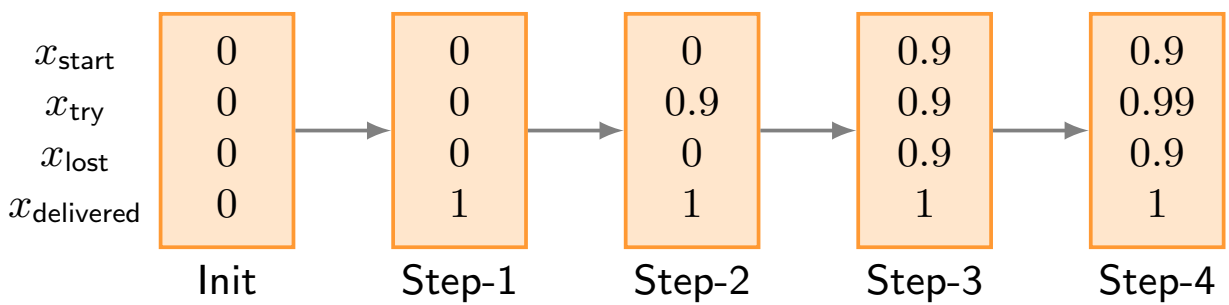

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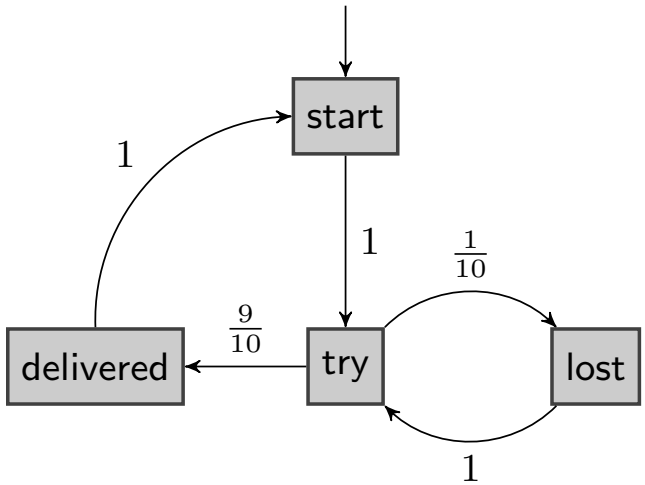

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Value Iteration :

	Init	Step-1	Step-2	Step-3	Step-4	Step-5
x_{start}	0	0	0	0.9	0.9	0.99
x_{try}	0	0	0.9	0.9	0.99	0.99
x_{lost}	0	0	0	0.9	0.9	0.99
$x_{\text{delivered}}$	0	1	1	1	1	1

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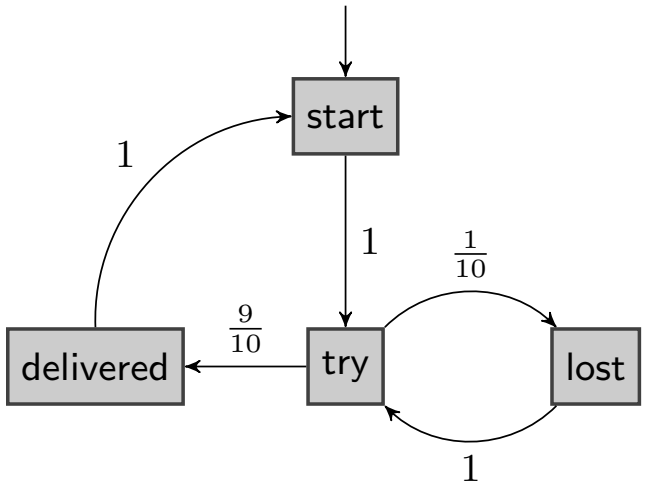

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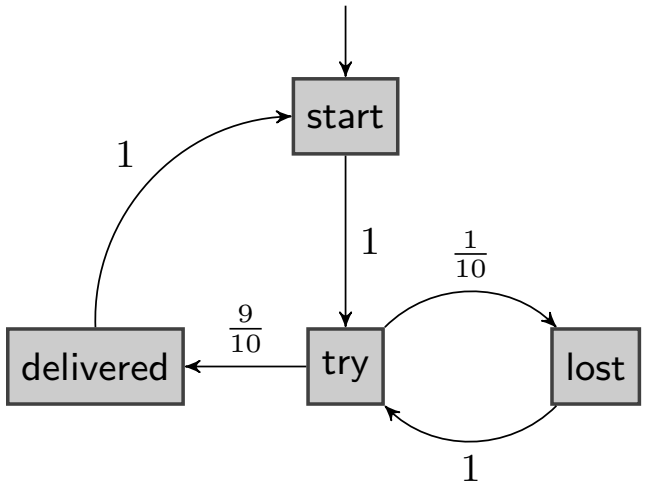

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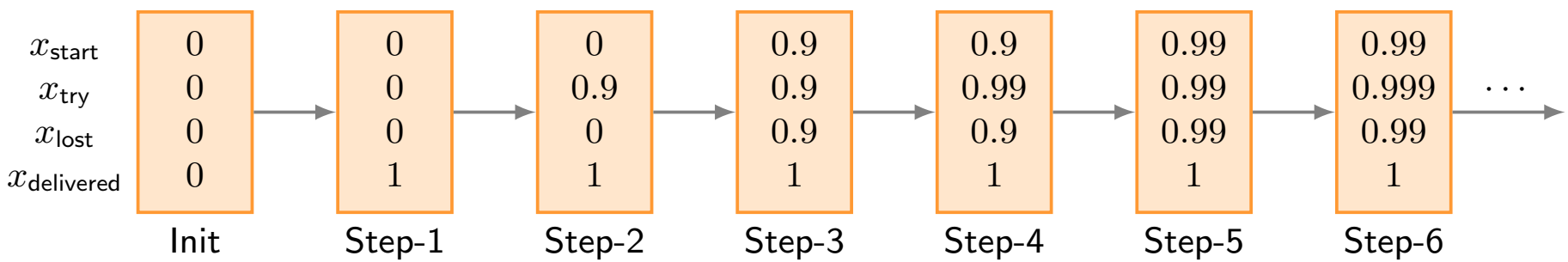

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More general setting :

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- Probabilistic Computation Tree Logic (PCTL) :
  - Probabilistic analogue of CTL
  - Probabilistic quantifier : $\mathbb{P}_{\bowtie p}(\cdot)$
  - Modal operators : $\mathcal{X}$ (next), $\mathcal{U}$ (until).

- Is the probability of 'delivered' without being 'lost' ≥ 0.575 ?

$$\mathbb{P}_{\geq 0.575}[\neg \text{lost}\, \mathcal{U}\, \text{delivered}]$$

- Is every message almost surely delivered?

$$\mathbb{P}_{=1}[\text{true}\, \mathcal{U}\, \text{delivered}]$$

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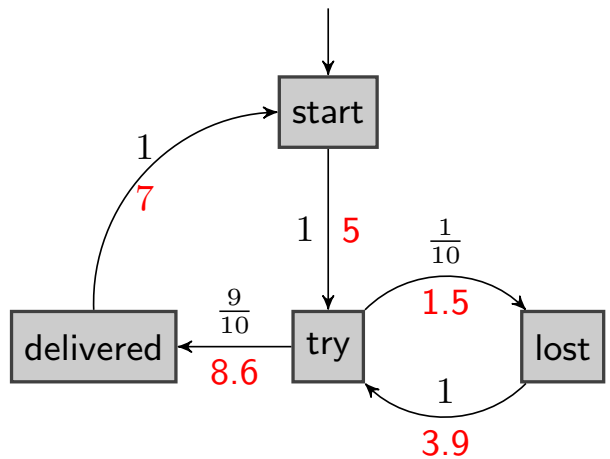

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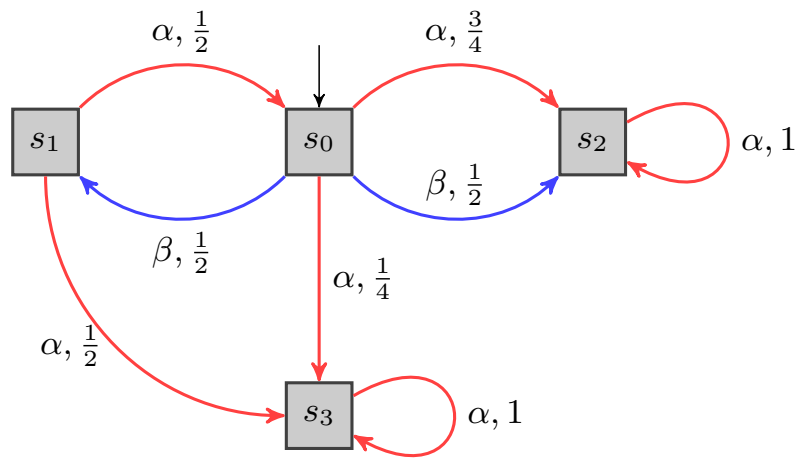
- Markov Decision Processes (MDPs)
  - Non-deterministic choice (actions)
  - Probability distribution for every action from a state



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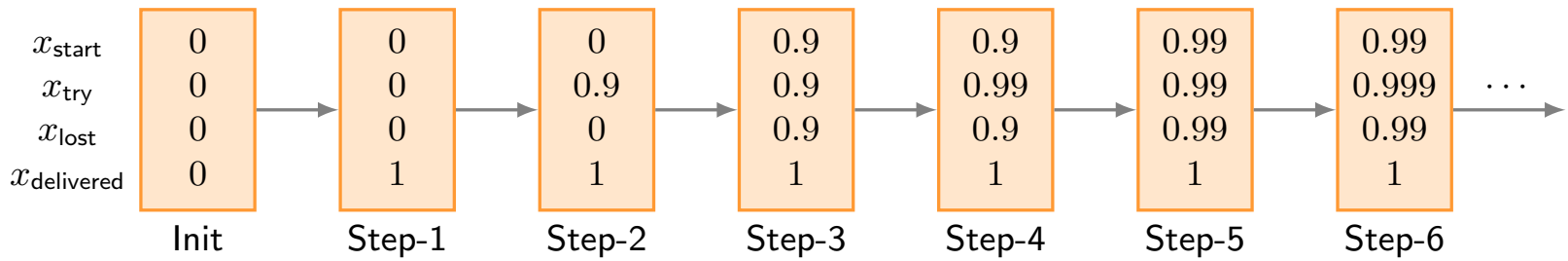

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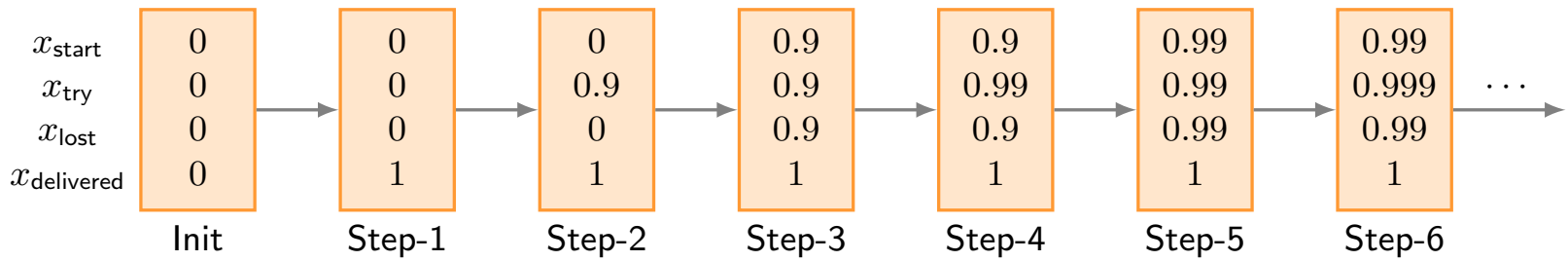

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$$\begin{pmatrix} x_{\text{start}} \\ x_{\text{try}} \\ x_{\text{lost}} \\ x_{\text{delivered}} \end{pmatrix} = \begin{pmatrix} 1-10^{-i} \\ 1-10^{-i} \\ 1-10^{-i} \\ 1 \end{pmatrix}$$

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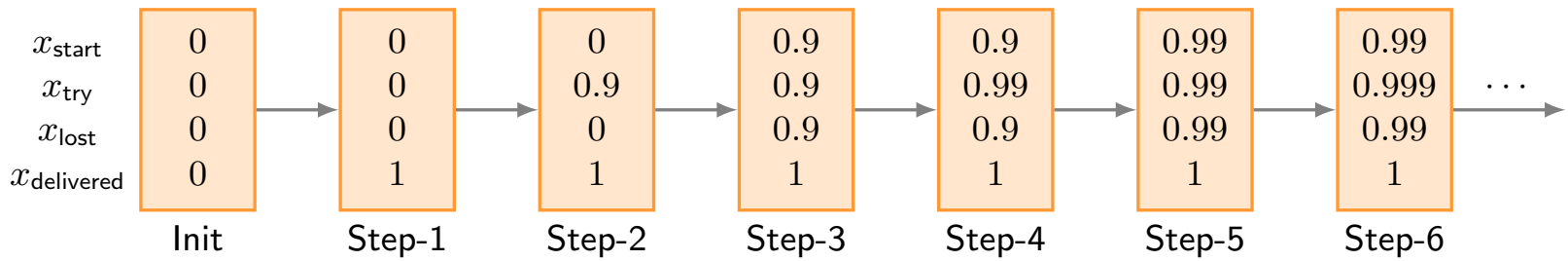

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Exact Quantitative Model Checking

- Approximate solution techniques can lead to unreliable results
- Incorrect analysis of systems
- Verification tools must strive to get the exact answers
- Existing techniques for exact model checking:

1. Linear programming
2. Parametric model checking
3. State Elimination

} Implemented in state-of-the-art quantitative model checkers - PRISM, STORM, etc.,

Rational Search : Key Ideas

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   - Fixpoint check : $\overline{x} = A\overline{x} + \overline{b}$
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Rational Search : Overview

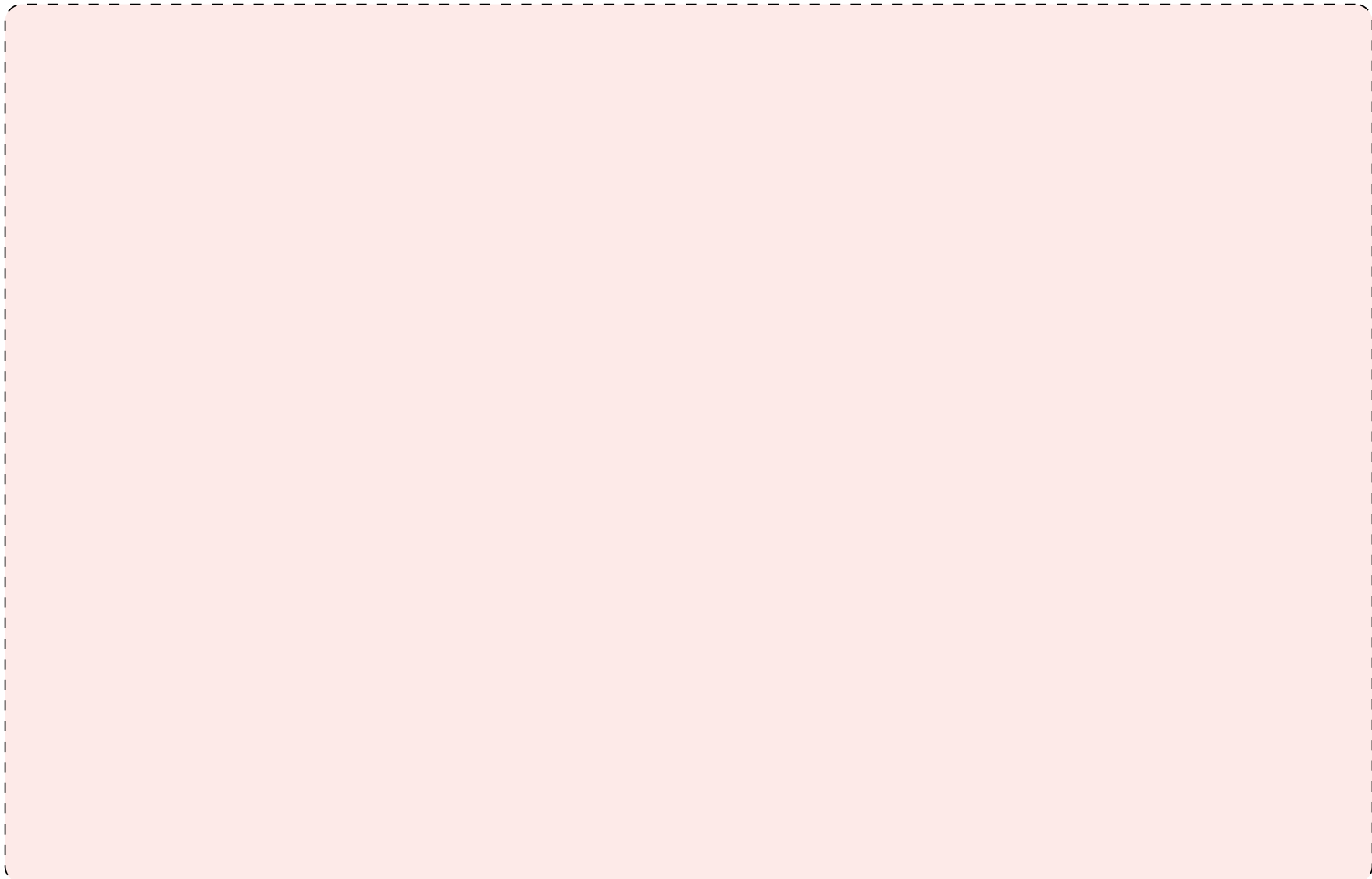

Figure: RationalSearch : Overview

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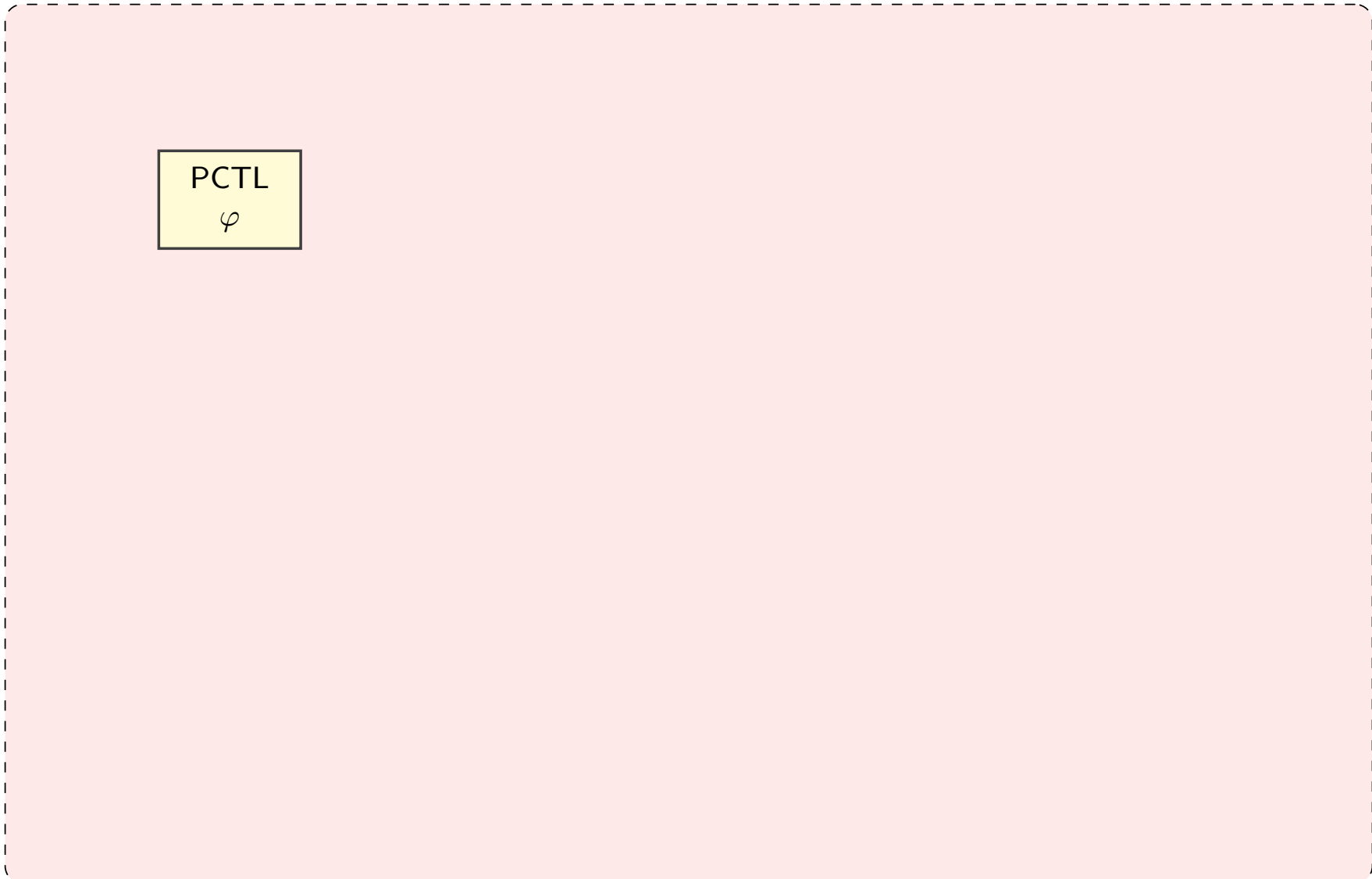

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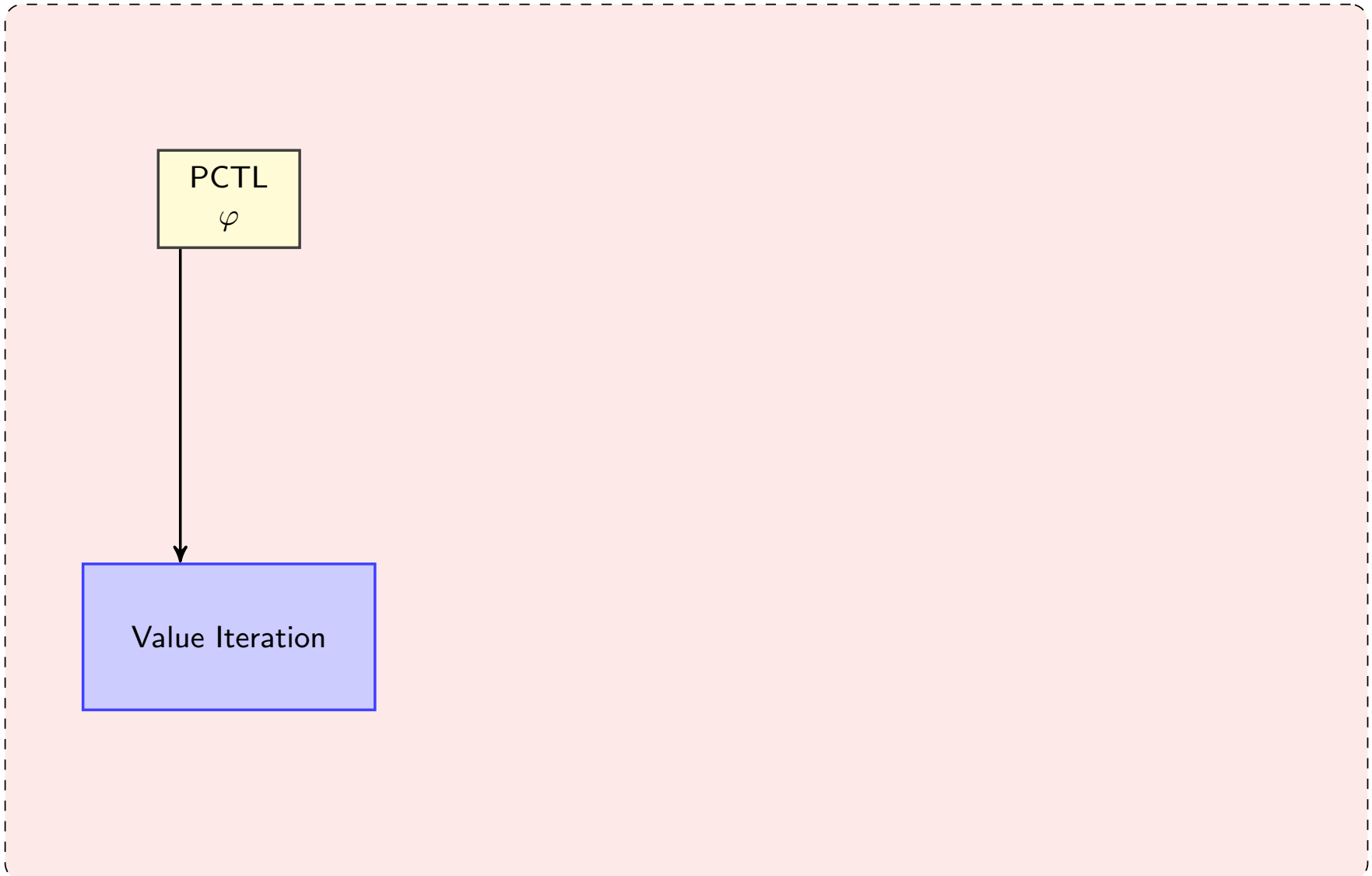

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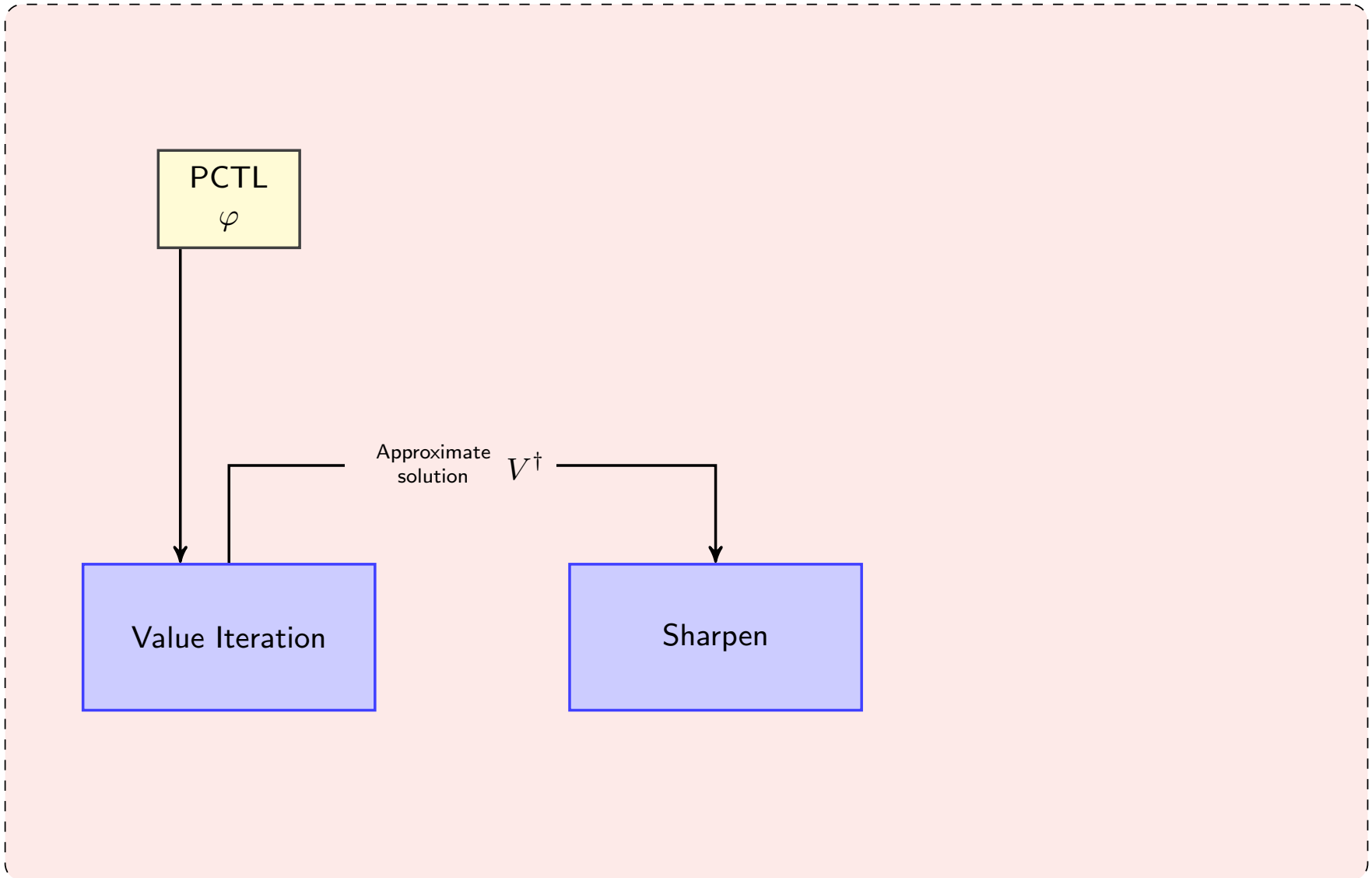

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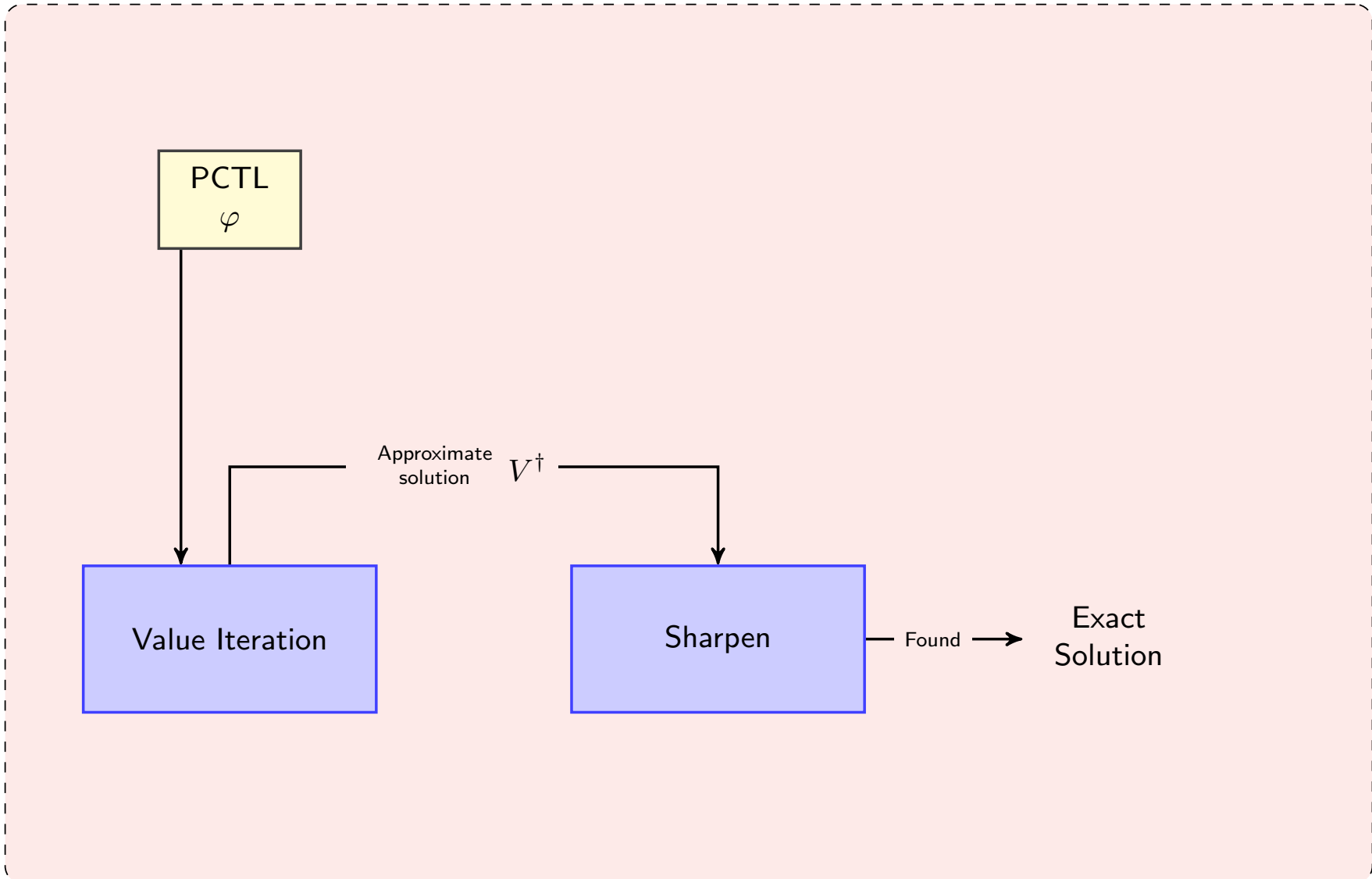

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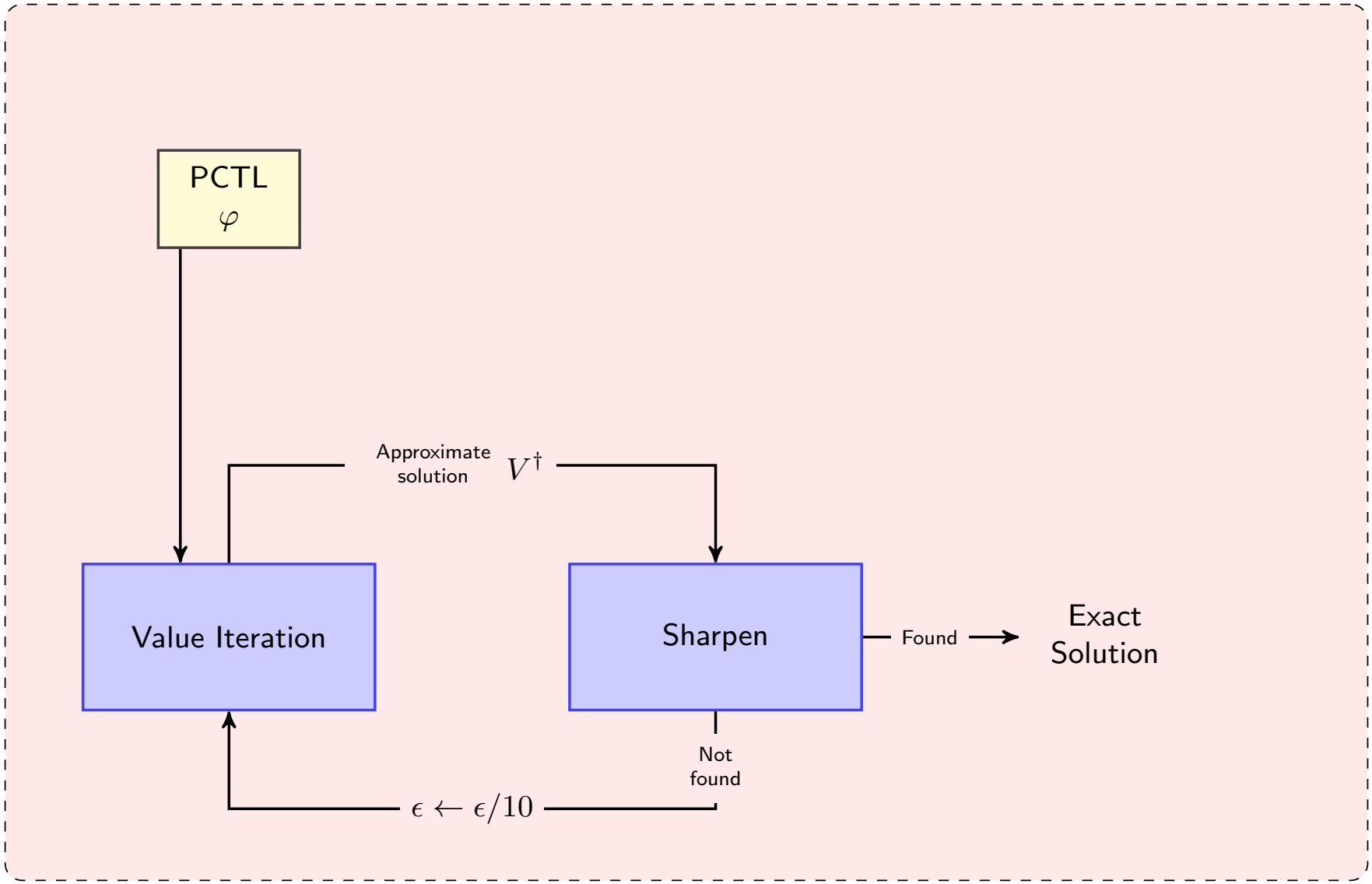

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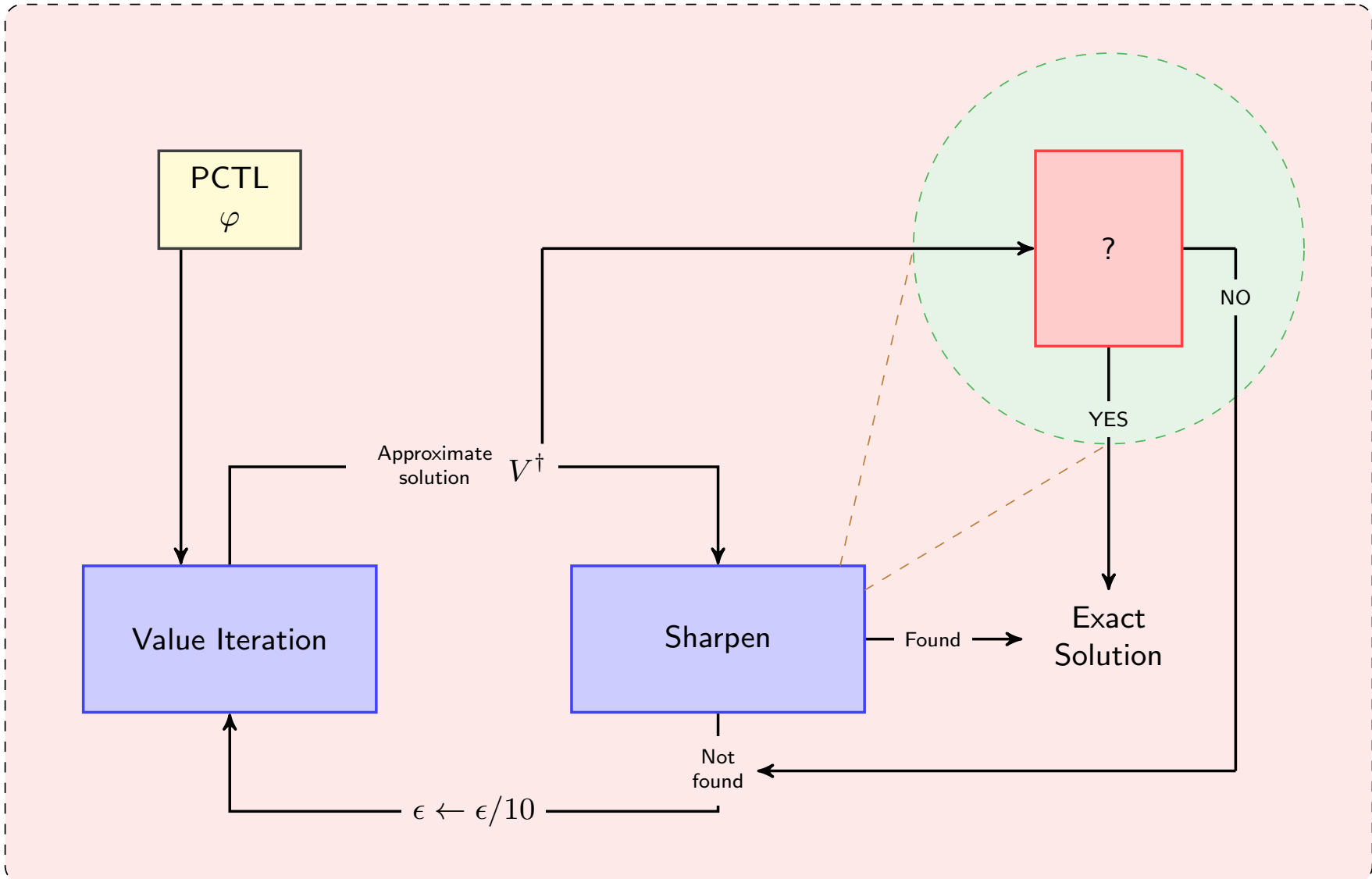

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Our Secret Ingredient : Kwek Mehlhorn Algorithm

For any interval $I = [\frac{\alpha}{\beta}, \frac{\gamma}{\delta}]$ with rational endpoints, there is a unique minimal rational $p_{\text{min}}/q_{\text{min}} \in I$ such that

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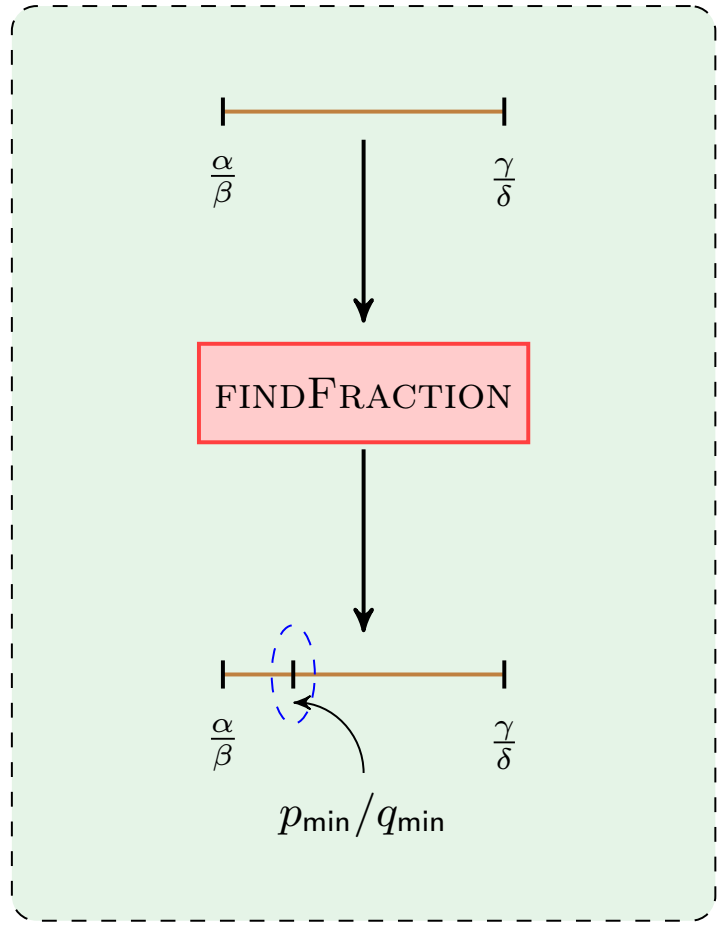

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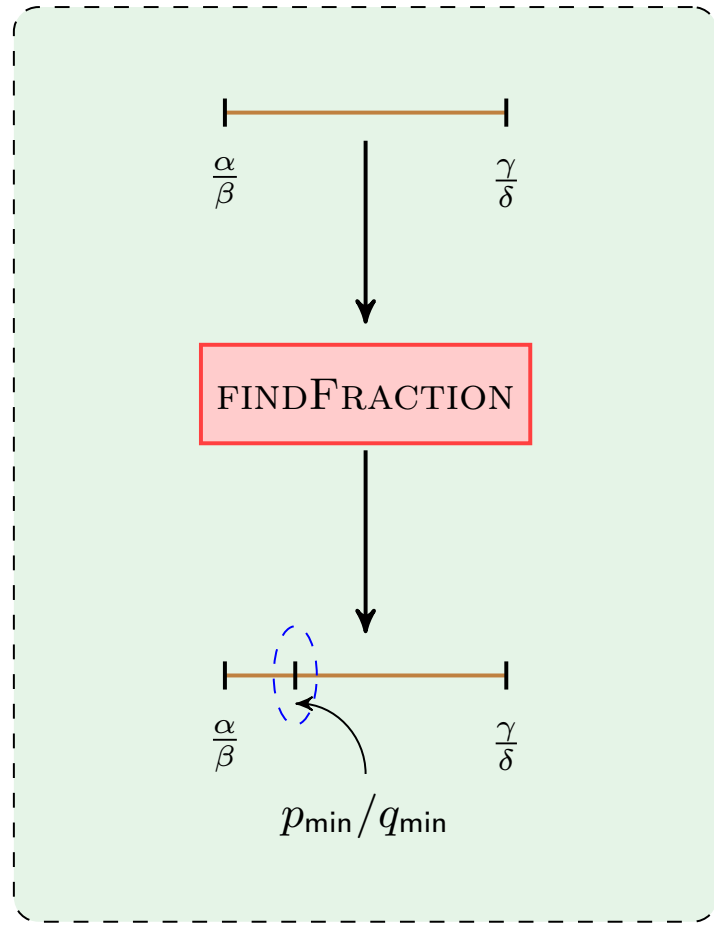

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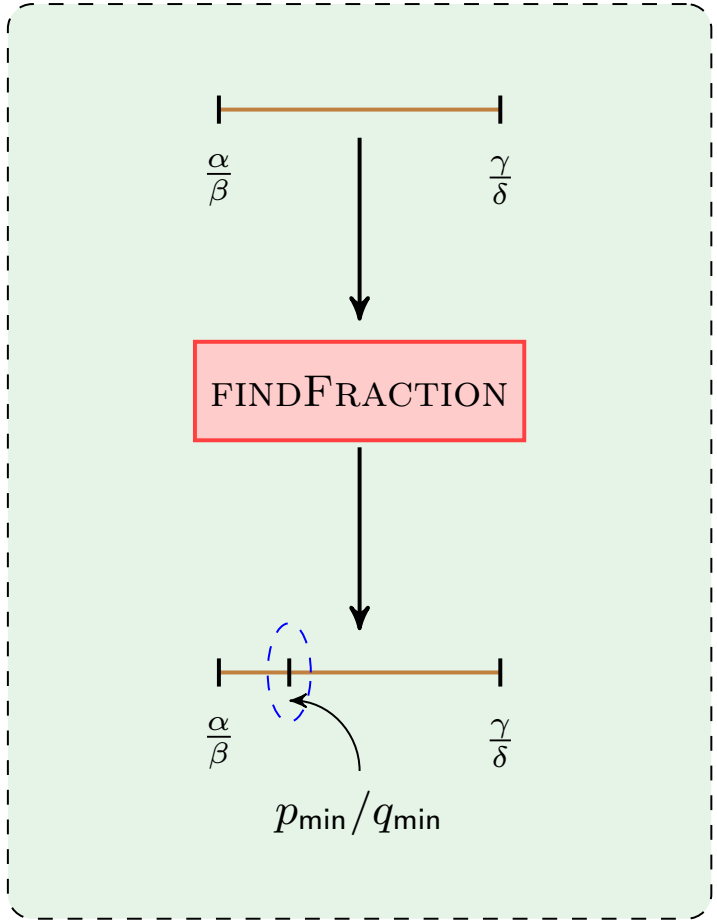

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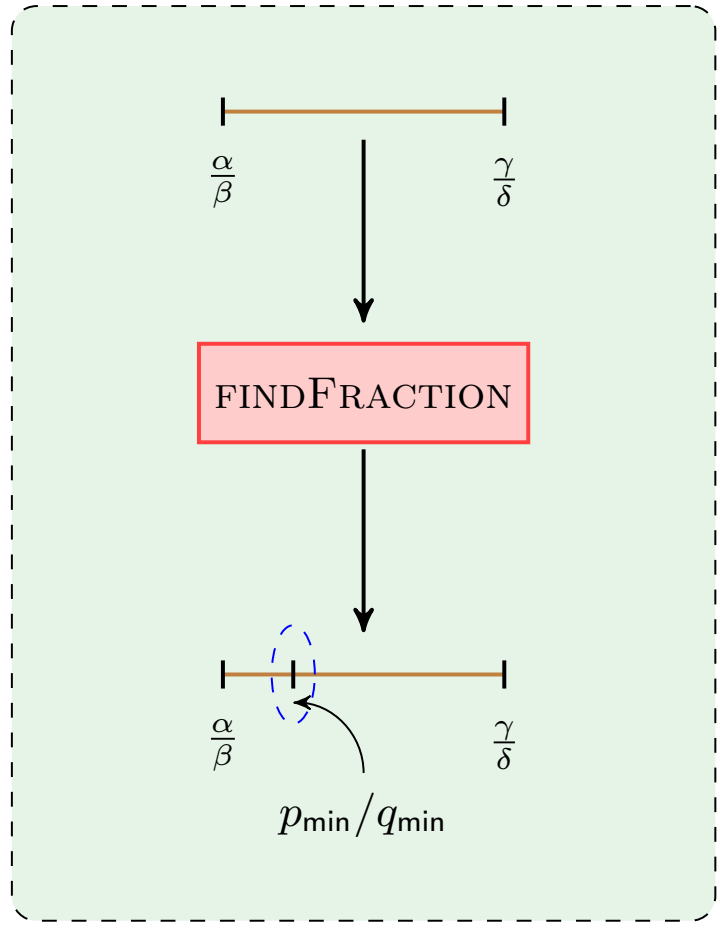

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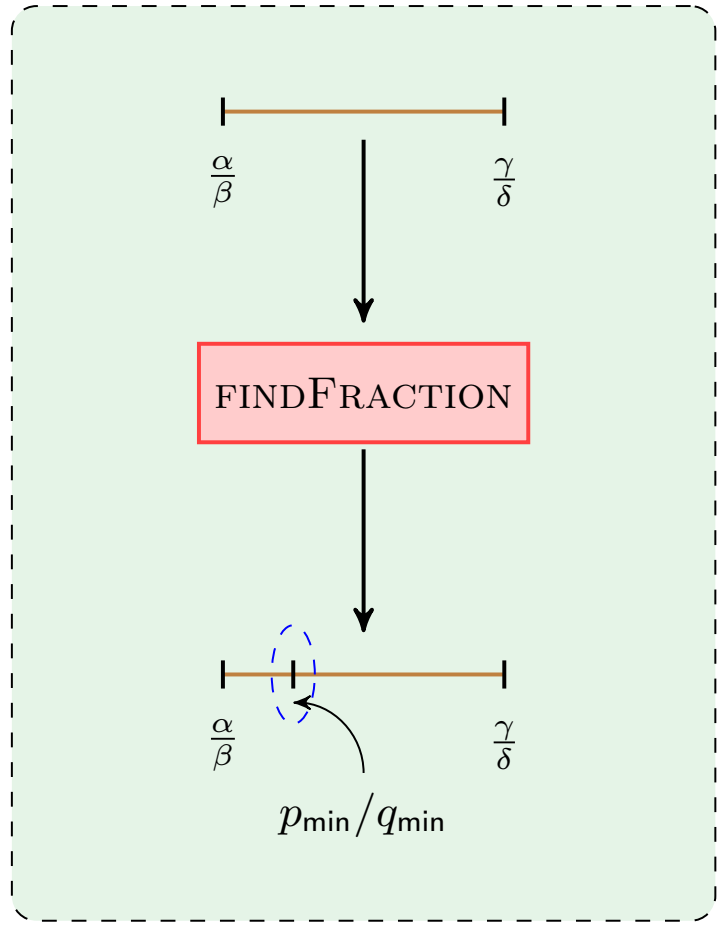

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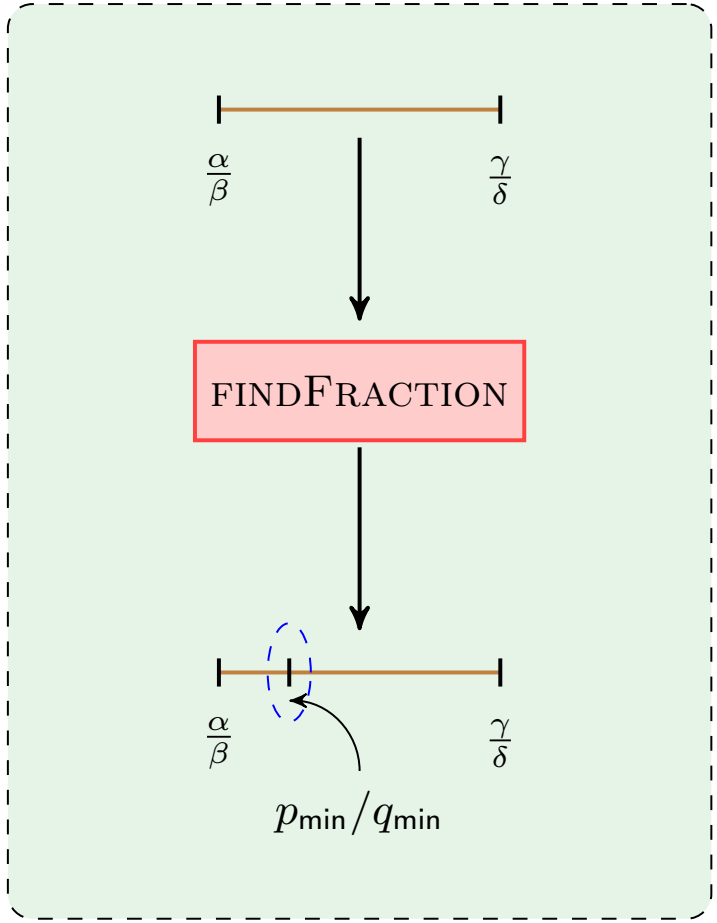
- Can be found efficiently (in $O(\log M)$ steps) due to Kwek, Mehlhorn et. al.



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Sharpening an approximation :

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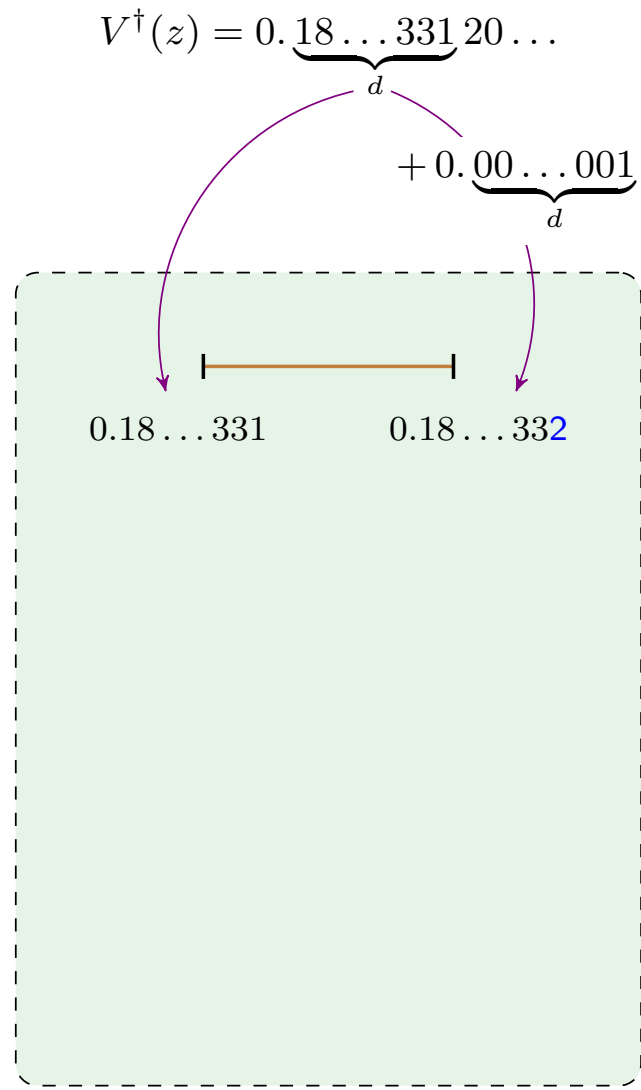

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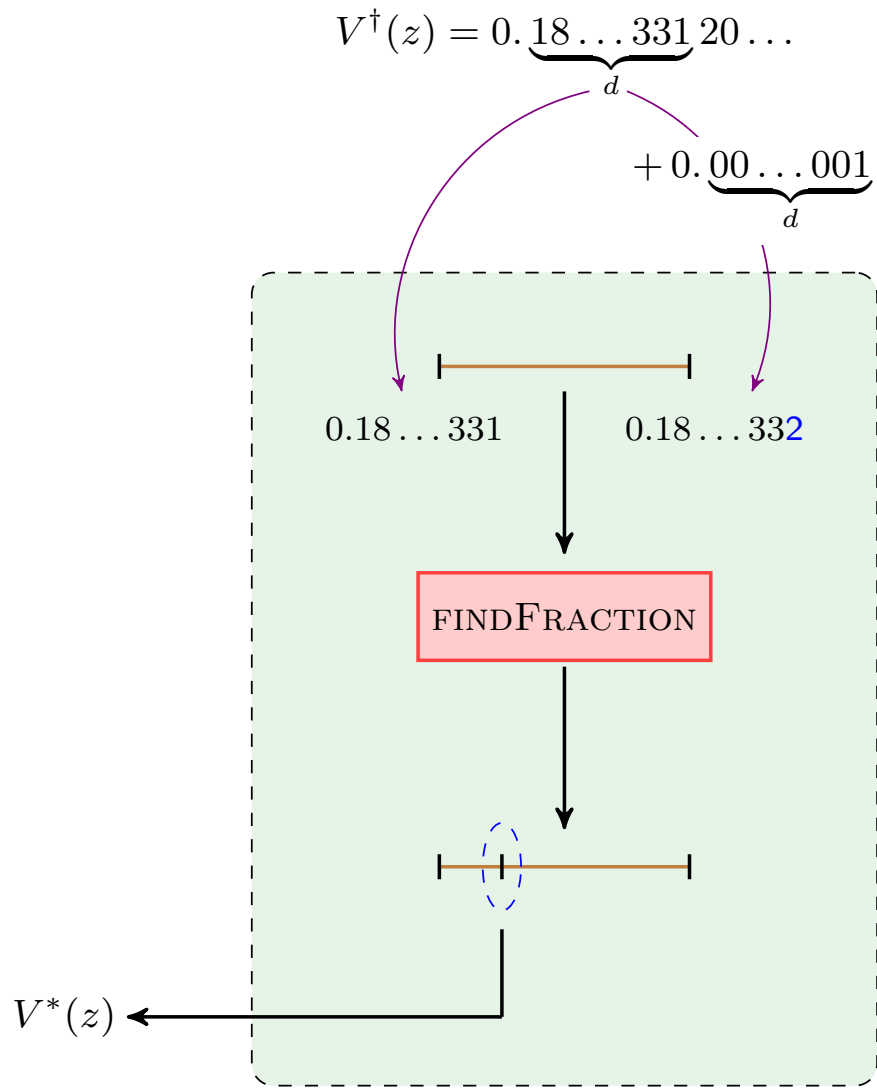
   - Compute minimal fraction in this interval, using Kwek Mehlhorn algorithm



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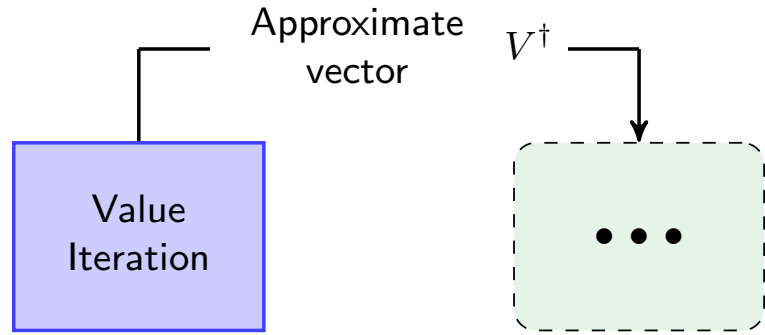

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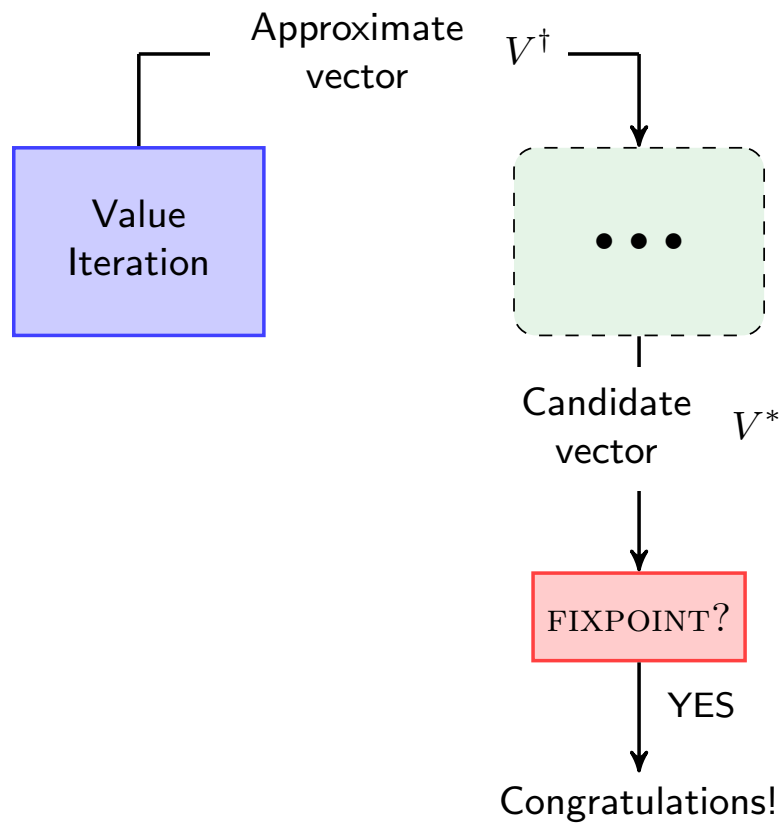

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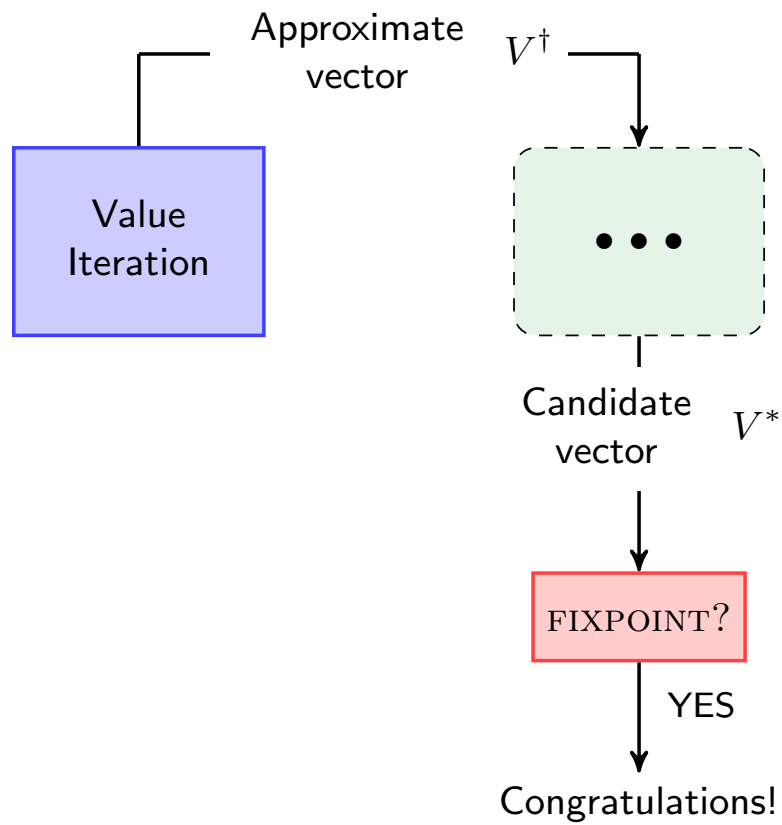

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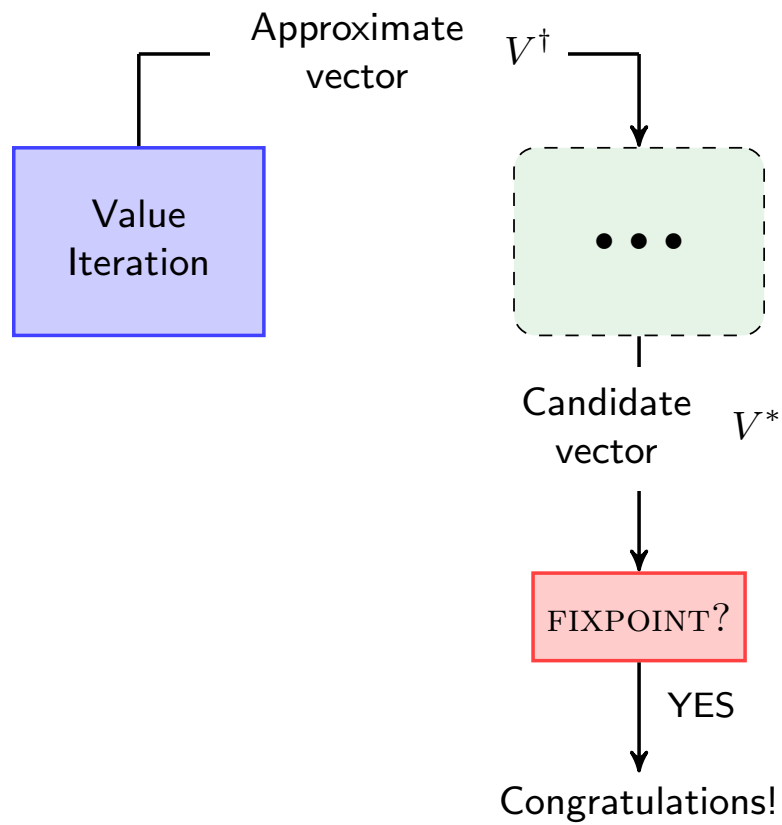

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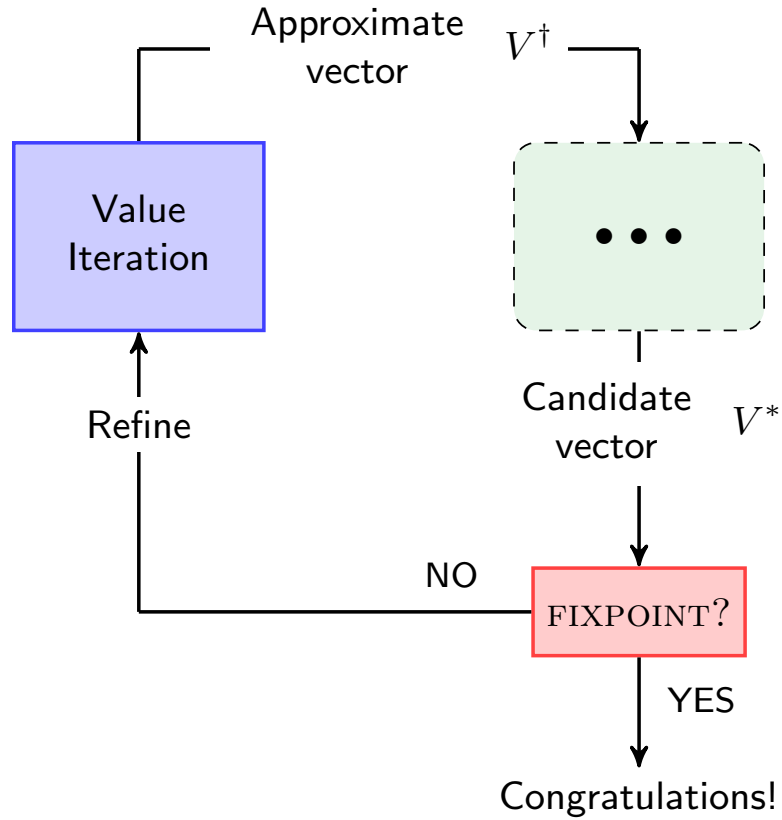

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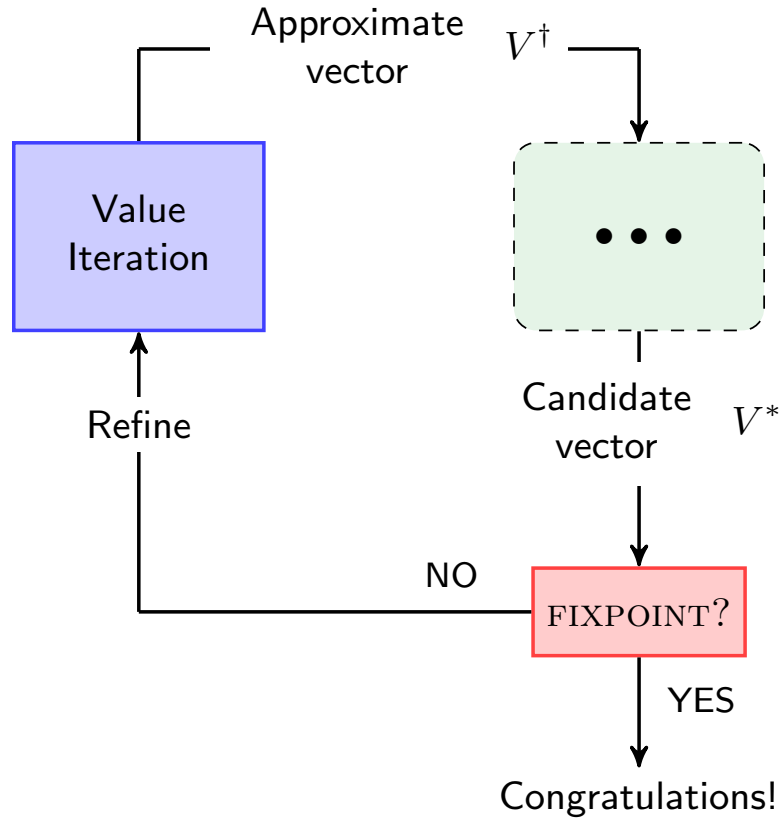

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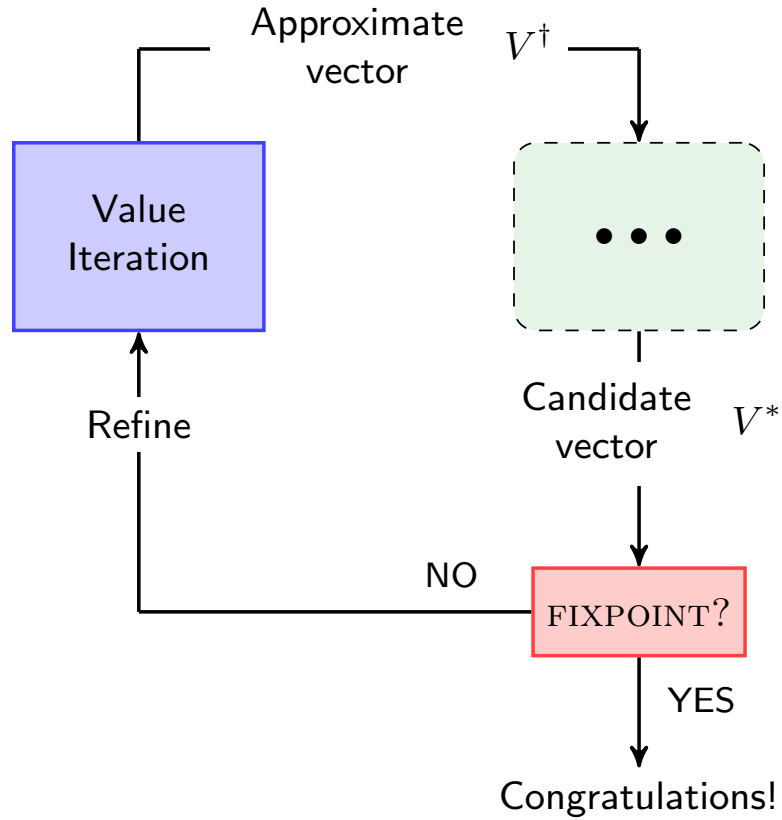
Details in the paper



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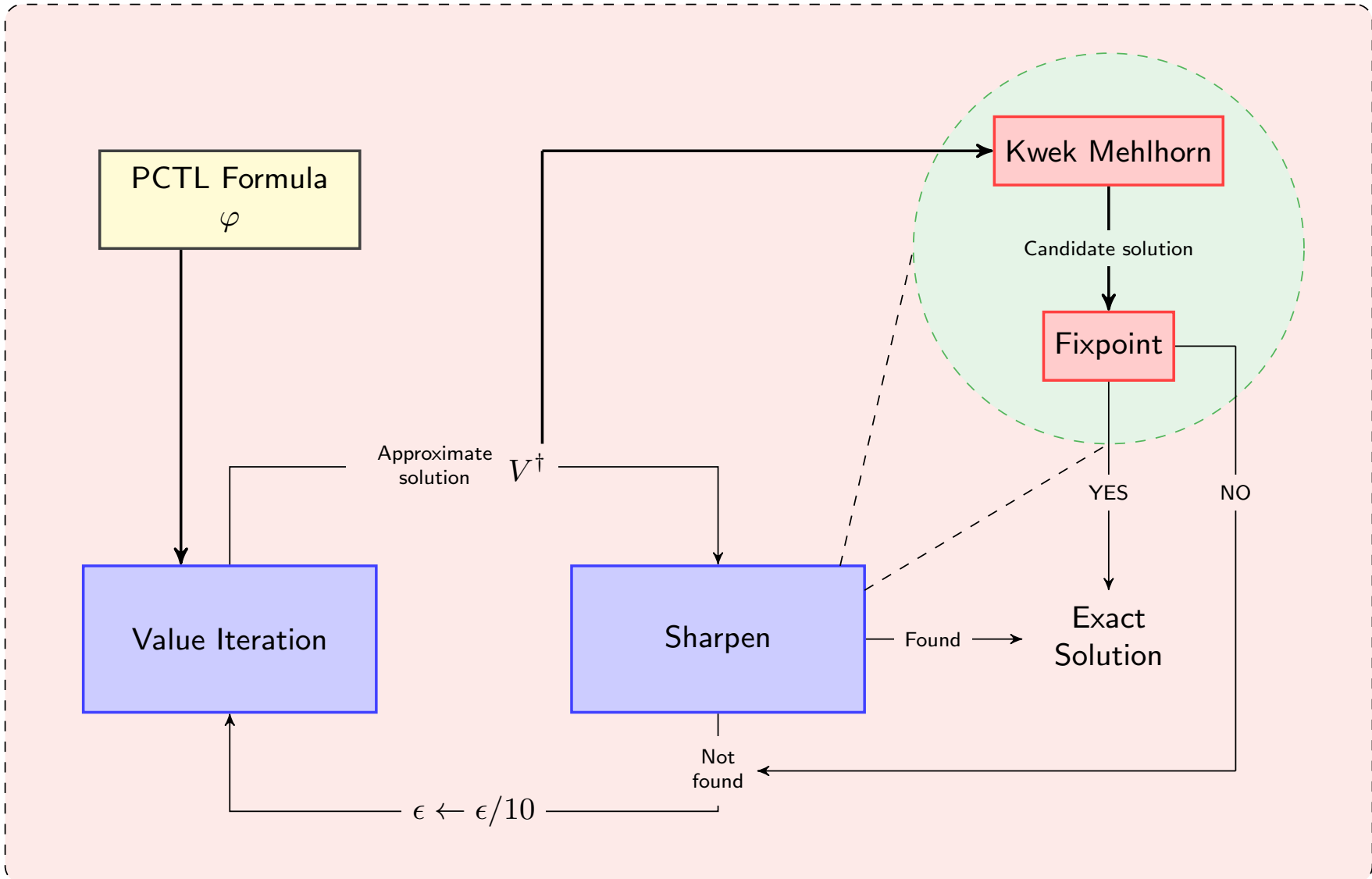
Rational Search : Recap



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– Tool RationalSearch, implemented on top of PRISM 

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Evaluation

1	2	3	4	5	6	7	8	9	10	11
Model			RationalSearch						PRISM	STORM
Name	Parameter	States	Explicit		MTBDD		Hybrid			
			Time (s)	Overhead (%)	Time (s)	Overhead (%)	Time (s)	Overhead (%)	Time (s)	Time (s)
Biased Coins	11	180K	23.1	336	0.125	179	0.178	225	1449.7	3.2
Dice	6	4.8M	OOM	N/A	1.8	2.1	6.5	12	TO	63
Din. Cryptographers	8	190K	18.9	197	0.278	70	0.364	105	356.2	2.4
Din. Philosophers	3	956	0.41	165	1.9	4.8	0.133	98	3.128	0.65
ECS	14	4.8M	OOM	N/A	2.4	23	11.6	79	TO	TO
Fair Exchange	400	320K	14.6	423	2.0	44	2.2	51	TO	1.1
Firewire	11,000	430K	122.2	225	15.1	0.2	19.5	21	232.3	29.5
Leader Election	4	12K	1.8	226	5.0	30	20.4	25	80	0.042
Virus Infection	3	809	0.5	165	2.8	52	0.17	93	0.98	0.032

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Dice	6	4.8M	OOM	N/A	1.8	2.1	6.5	12	TO	63
Din. Cryptographers	8	190K	18.9	197	0.278	70	0.364	105	356.2	2.4
Din. Philosophers	3	956	0.41	165	1.9	4.8	0.133	98	3.128	0.65
ECS	14	4.8M	OOM	N/A	2.4	23	11.6	79	TO	TO
Fair Exchange	400	320K	14.6	423	2.0	44	2.2	51	TO	1.1
Firewire	11,000	430K	122.2	225	15.1	0.2	19.5	21	232.3	29.5
Leader Election	4	12K	1.8	226	5.0	30	20.4	25	80	0.042
Virus Infection	3	809	0.5	165	2.8	52	0.17	93	0.98	0.032

Figure: Experimental Evaluation

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Thank You !

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